Linear Algebra Practice Final
Spring 2019

1. (10 points) Let

$$A = \begin{pmatrix} 1 & 3 & 0 \\ 0 & -1 & 2 \end{pmatrix}$$

Compute AA^T and determine if it is diagonalizable. Justify your answer either with a clear demonstration or with reference to a relevant theorem.

2. Using the same matrix A as in problem (1),

$$A = \begin{pmatrix} 1 & 3 & 0 \\ 0 & -1 & 2 \end{pmatrix}$$

answer the following questions:

(a) (5 points) What is $\det(A^T A)$?

(b) (5 points) Use the above result to determine if the columns of $A^T A$ are linearly independent.

3. Consider the matrices $A = \begin{pmatrix} 2 & 1 \\ 0 & 2 \end{pmatrix}$ and $B = \begin{pmatrix} 2 & 0 \\ 0 & 2 \end{pmatrix}$.

(a) (5 points) Find the eigenvalues of each matrix.

(b) (5 points) Determine the geometric multiplicities of each eigenvalue, for each matrix. Which of A and B is diagonalizable?

(c) (5 points) Determine whether the two matrices are similar, i.e. whether $B = PAP^{-1}$ for some $P \in \mathrm{GL}(2, \mathbb{R})$.

4. Consider the vector $\mathbf{v} = \begin{pmatrix} 5 \\ 2 \end{pmatrix}$ in $\mathbb{R}^2$.

(a) (5 points) Determine $[\mathbf{v}]_\beta$ if $\beta = \left(\begin{pmatrix} 1 \\ 2 \end{pmatrix}, \begin{pmatrix} 3 \\ 4 \end{pmatrix} \right)$.

(b) (5 points) Find the standard representation of the projection $\pi_{\mathbf{w}}$ onto the vector $\mathbf{w} = R_{\pi/2}\mathbf{v}$ orthogonal to $\mathbf{v}$.

(c) (5 points) Find the standard representation of the reflection R_ℓ across $\ell = \text{span}(\mathbf{v})$.

5. Let

$$A = \begin{pmatrix} 2 & 3 \\ 3 & -6 \end{pmatrix}$$

(a) (5 points) Find the eigenvalues of A.

(b) (5 points) Find corresponding eigenvectors of A.

(c) (5 points) Is A *orthogonally* diagonalizable? Justify your answer with a clear demonstration or with reference to a relevant theorem.

(d) (5 points) Let

$$D = [A]_\beta = \begin{pmatrix} \lambda_1 & 0 \\ 0 & \lambda_2 \end{pmatrix}$$

be the diagonal matrix with the eigenvalues of A, λ_1 and λ_2, you got in part (b), whose corresponding eigenvectors $\mathbf{v}_1$ and $\mathbf{v}_2$ you got in part (c). Use the fact that $A = PDP^{-1}$, and therefore $e^A = e^{PDP^{-1}} = Pe^D P^{-1}$, to compute e^A.

Here, of course, $e^A \stackrel{\text{def}}{=} \sum_{n=0}^{\infty} \frac{1}{n!} A^n$.

6. (5 points) Show that $A = \begin{pmatrix} 1 & 2-i \\ 2+i & 3 \end{pmatrix}$ is Hermitian.

7. (5 points) Let $A \in \mathrm{O}(n)$ be an $n \times n$ orthogonal matrix. Show that

$$(A\mathbf{x}) \cdot (A\mathbf{y}) = \mathbf{x} \cdot \mathbf{y}$$

for all $\mathbf{x}, \mathbf{y} \in \mathbb{R}^n$. *Hint: the dot product* $\mathbf{x} \cdot \mathbf{y}$ *can be computed in terms of matrices,*

$$\mathbf{x} \cdot \mathbf{y} = \begin{pmatrix} x_1 & \cdots & x_n \end{pmatrix} \begin{pmatrix} y_1 \\ \vdots \\ y_n \end{pmatrix} = \mathbf{x}^T \mathbf{y}$$

8. (5 points) Show that for all $\mathbf{x}, \mathbf{y} \in \mathbb{R}^n$, we have

$$\mathbf{x} \cdot \mathbf{y} = \frac{1}{4}\|\mathbf{x} + \mathbf{y}\|^2 - \frac{1}{4}\|\mathbf{x} - \mathbf{y}\|^2$$

This is the **polar identity**.[1]

[1] There is also a complex polar identity in $\mathbb{C}^n$, in terms of the Hermitian inner product,

$$\mathbf{z} \cdot \mathbf{w} = \frac{1}{4}\sum_{k=1}^{4} \|\mathbf{z} + i^k \mathbf{w}\|^2$$

which it would be a worthwhile exercise to try to prove.

9. (10 points) Determine if $\mathbf{w}$ lies in the span of $\mathbf{u}$ and $\mathbf{v}$. If so, write $\mathbf{w}$ as a linear combination of $\mathbf{u}$ and $\mathbf{v}$.

$$\mathbf{u} = \begin{pmatrix} 0 \\ 2 \\ -1 \end{pmatrix}, \quad \mathbf{v} = \begin{pmatrix} 2 \\ 2 \\ 0 \end{pmatrix}, \quad \mathbf{w} = \begin{pmatrix} 6 \\ 16 \\ -5 \end{pmatrix}$$

10. (10 points) Use the following properties of the determinant,

- $\det(AB) = (\det A)(\det B)$
- $\det(A^T) = \det A$

to prove that for any square matrix $A \in M_n(\mathbb{R})$ we have

$$|\det A| = \sqrt{\det(A^T A)}$$

The RHS of this equation is sometimes taken for the definition of the volume of the parallelpiped spanned by the columns of A. What's even nicer, this also works for non-square $m \times n$ matrices, with $m > n$. For example, a 3×2 matrix with two columns of size 3, would give the *area* of the parallelogram spanned by its columns *in* $\mathbb{R}^3$.

11. (10 points) The **trace** of a square matrix $A \in M_n(\mathbb{R})$ is the sum of its diagonal entries,

$$\operatorname{tr} A = \sum_{i=1}^{n} a_{ii}$$

and we know that $\operatorname{tr}(AB) = \operatorname{tr}(BA)$. It follows from this that if A is diagonalizable, then $\operatorname{tr} A = \operatorname{tr} D = \sum_{i=1}^{n} \lambda_i$. Use this and the fact that any real or complex matrix is diagonalizable over $\mathbb{C}$ to show that

$$\det e^A = e^{\operatorname{tr} A}$$

[*Hint: use the diagonalizability of* A *in combination with the exponential,* $e^A = e^{PDP^{-1}} = Pe^D P^{-1}$.]

12. (5 points) **True or False**: If $A \in \mathrm{Skew}_n(\mathbb{R})$, then $\operatorname{tr} A = 0$.

13. (5 points) **True or False**: If A is diagonalizable with characteristic polynomial factoring as

$$c_A(\lambda) = \det(A - \lambda I) = (-1)^n(\lambda - \lambda_1)^{n_1}\lambda - \lambda_2)^{n_2} \cdots (\lambda - \lambda_k)^{n_k}$$

where $\lambda_1, \ldots, \lambda_k$ are A's distinct eigenvalues, then

$$\det A = \lambda_1^{n_1}\lambda_2^{n_2} \cdots \lambda_k^{n_k}$$

14. (5 points) **True or False**: The standard representation of a reflection is always an orthogonal matrix.

15. (5 points) **True or False**: If A is diagonalizable with distinct eigenvalues $\lambda_1, \ldots, \lambda_k$, then we may write A as a sum of projections onto the corresponding eigenspaces,

$$A = \lambda_1 \pi_{E_{\lambda_1}} + \lambda_2 \pi_{E_{\lambda_2}} + \cdots + \lambda_k \pi_{E_{\lambda_k}}$$

Here, of course, and only to round things off, we mention that

$$\begin{aligned} \pi_{E_{\lambda_i}}(\mathbf{x}) &= \pi_{\mathbf{v}_1}(\mathbf{x}) + \cdots + \pi_{\mathbf{v}_k}(\mathbf{x}) \\ &= \frac{\mathbf{v}_1 \cdot \mathbf{x}}{\|\mathbf{v}_1\|^2}\mathbf{v}_1 + \cdots + \frac{\mathbf{v}_{n_i} \cdot \mathbf{x}}{\|\mathbf{v}_{n_i}\|^2}\mathbf{v}_{n_i} \end{aligned}$$

where $\mathbf{v}_1, \ldots, \mathbf{v}_{n_i}$ are the basis eigenvectors for E_{λ_i}.

16. (5 points) Show that if U and V are two subpsaces of $\mathbb{C}^n$, then so is their sum

$$U + V \stackrel{\text{def}}{=} \{\mathbf{u} + \mathbf{v} \mid \mathbf{u} \in U \text{ and } \mathbf{v} \in V\}$$