Linear Algebra Practice Midterm 2
Spring 2019

1. (5 points) State the definition of a basis β for a vector space V.

A basis β is a collection of vectors in V which are both spanning ($V = \text{span}\,\beta$) and linearly independent.

2. (a) (5 points) Show directly from the definition that the vectors $\mathbf{u} = \begin{pmatrix} 1 \\ 4 \end{pmatrix}$ and $\mathbf{v} = \begin{pmatrix} 5 \\ 6 \end{pmatrix}$ form a basis $\beta = (\mathbf{u}, \mathbf{v})$ for $\mathbb{R}^2$.

(i) To see linear independence, note $a\vec{u} + b\vec{v} = \vec{0}$

$$\Rightarrow \begin{pmatrix} | & | \\ \vec{u} & \vec{v} \\ | & | \end{pmatrix}\begin{pmatrix} a \\ b \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \end{pmatrix} \longrightarrow \begin{pmatrix} 1 & 5 \\ 4 & 6 \end{pmatrix}\begin{pmatrix} a \\ b \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \end{pmatrix}$$

$$\longrightarrow \begin{pmatrix} 1 & 5 \\ 0 & -14 \end{pmatrix}\begin{pmatrix} a \\ b \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \end{pmatrix} \Rightarrow a = b = 0.$$

(ii) For spanning: let $\vec{x} \in \mathbb{R}^2$, & find a, b s.t. $a\vec{u} + b\vec{v} = \vec{x}$

$$\Rightarrow \begin{pmatrix} | & | \\ \vec{u} & \vec{v} \\ | & | \end{pmatrix}\begin{pmatrix} a \\ b \end{pmatrix} = \vec{x}$$

$$\Rightarrow \begin{pmatrix} 1 & 5 \\ 4 & 6 \end{pmatrix}\begin{pmatrix} a \\ b \end{pmatrix} = \begin{pmatrix} x \\ y \end{pmatrix}$$

$$\Rightarrow \boxed{\begin{pmatrix} a \\ b \end{pmatrix} = \frac{-1}{14}\begin{pmatrix} 6 & -5 \\ -4 & 1 \end{pmatrix}\begin{pmatrix} x \\ y \end{pmatrix}}$$

will get you there!

(b) (5 points) Is the basis $\beta = (\mathbf{u}, \mathbf{v})$ positively oriented?

$$\det M_{\beta\sigma} = \det\begin{pmatrix} | & | \\ \vec{u} & \vec{v} \\ | & | \end{pmatrix} = \det\begin{pmatrix} 1 & 5 \\ 4 & 6 \end{pmatrix} = -14 < 0$$

No, negatively oriented.

(c) (5 points) Find the β-representation of $\mathbf{x} = \begin{pmatrix} 1 \\ 1 \end{pmatrix}$.

$$[\vec{x}]_\beta = M_{\sigma\beta} [\vec{x}]_\sigma$$

$$= M_{\beta\sigma}^{-1} \vec{x}$$

$$= \frac{-1}{14} \begin{pmatrix} 6 & -5 \\ -4 & 1 \end{pmatrix} \begin{pmatrix} 1 \\ 1 \end{pmatrix}$$

$$= \boxed{\frac{-1}{14} \begin{pmatrix} 1 \\ -3 \end{pmatrix}}$$

or

$$\boxed{\frac{1}{14} \begin{pmatrix} -1 \\ 3 \end{pmatrix}}$$

(d) (5 points) Verify that your answer in part (c) is correct: if $[\mathbf{x}]_\beta = \begin{pmatrix} a \\ b \end{pmatrix}$, verify that $\mathbf{x} = a\mathbf{u} + b\mathbf{v}$.

$$\frac{-1}{14} \vec{u} + \frac{3}{14} \vec{v} = \frac{-1}{14} \begin{pmatrix} 1 \\ 4 \end{pmatrix} + \frac{3}{14} \begin{pmatrix} 5 \\ 6 \end{pmatrix}$$

$$= \begin{pmatrix} -1/14 + 15/14 \\ -4/14 + 18/14 \end{pmatrix}$$

$$= \begin{pmatrix} 14/14 \\ 14/14 \end{pmatrix}$$

$$= \begin{pmatrix} 1 \\ 1 \end{pmatrix} \checkmark$$

$$= \vec{x}$$

3. Consider the vector $\mathbf{v} = \begin{pmatrix} 1 \\ 2 \end{pmatrix}$ in $\mathbb{R}^2$.

(a) (5 points) Find the standard representation of the projection $\pi_{\mathbf{v}}$ onto $\mathbf{v}$.

$$[\pi_{\vec{v}}]_\sigma = \begin{pmatrix} | & | \\ [\pi_{\vec{v}}(\vec{e}_1)]_\sigma & [\pi_{\vec{v}}(\vec{e}_2)]_\sigma \\ | & | \end{pmatrix}$$

$$= \begin{pmatrix} | & | \\ \frac{\vec{v}\cdot\vec{e}_1}{\|\vec{v}\|^2}\vec{v} & \frac{\vec{v}\cdot\vec{e}_2}{\|\vec{v}\|^2}\vec{v} \\ | & | \end{pmatrix}$$

$$= \begin{pmatrix} \left(\frac{\begin{pmatrix}1\\2\end{pmatrix}\cdot\begin{pmatrix}1\\0\end{pmatrix}}{1^2+2^2}\right)\begin{pmatrix}1\\2\end{pmatrix} & \left(\frac{\begin{pmatrix}1\\2\end{pmatrix}\cdot\begin{pmatrix}0\\1\end{pmatrix}}{1^2+2^2}\right)\begin{pmatrix}1\\2\end{pmatrix} \end{pmatrix}$$

$$= \boxed{\frac{1}{5}\begin{pmatrix} 1 & 2 \\ 2 & 4 \end{pmatrix}}$$

(b) (5 points) Find the standard representation of the projection $\pi_{\mathbf{w}}$ onto the vector $\mathbf{w} = R_{\pi/2}\mathbf{v}$ orthogonal to $\mathbf{v}$, .

$$\vec{\omega} = R_{\pi/2}\vec{v} = \begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix}\begin{pmatrix}1\\2\end{pmatrix} = \begin{pmatrix}-2\\1\end{pmatrix}$$

$$\Longrightarrow [\pi_{\vec{\omega}}]_\sigma = \begin{pmatrix} | & | \\ [\pi_{\vec{\omega}}(\vec{e}_1)]_\sigma & [\pi_{\vec{\omega}}(\vec{e}_2)]_\sigma \\ | & | \end{pmatrix}$$

$$= \begin{pmatrix} \frac{\vec{\omega}\cdot\vec{e}_1}{\|\vec{\omega}\|^2}\vec{\omega} & \frac{\vec{\omega}\cdot\vec{e}_2}{\|\vec{\omega}\|^2}\vec{\omega} \end{pmatrix}$$

$$= \begin{pmatrix} \frac{\begin{pmatrix}-2\\1\end{pmatrix}\cdot\begin{pmatrix}1\\0\end{pmatrix}}{(-2)^2+1^2}\begin{pmatrix}-2\\1\end{pmatrix} & \frac{\begin{pmatrix}-2\\1\end{pmatrix}\cdot\begin{pmatrix}0\\1\end{pmatrix}}{(-2)^2+1^2}\begin{pmatrix}-2\\1\end{pmatrix} \end{pmatrix}$$

$$= \boxed{\frac{1}{5}\begin{pmatrix} 4 & -2 \\ -2 & 1 \end{pmatrix}}$$

(c) (5 points) Find the standard representation of the reflection R_ℓ across $\ell = \text{span}(\mathbf{v})$.

Since $R_\ell(\vec{v}) = \vec{v}$ & $R_\ell \vec{\omega} = -\vec{\omega}$, if $\beta = (\vec{v}, \vec{\omega})$,

$$[R_\ell]_\beta = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}, \quad \text{so}$$

$$[R_\ell]_\sigma = M_{\beta\sigma} [R_\ell]_\beta M_{\beta\sigma}^{-1}$$

$$= \begin{pmatrix} | & | \\ \vec{v} & \vec{\omega} \\ | & | \end{pmatrix} \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix} \begin{pmatrix} | & | \\ \vec{v} & \vec{\omega} \\ | & | \end{pmatrix}^{-1}$$

$$= \underbrace{\begin{pmatrix} 1 & -2 \\ 2 & 1 \end{pmatrix} \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}} \underbrace{\left(\frac{-1}{5} \begin{pmatrix} 1 & 2 \\ -2 & 1 \end{pmatrix} \right)}$$

$$= \frac{-1}{5} \begin{pmatrix} 1 & 2 \\ 2 & -1 \end{pmatrix} \begin{pmatrix} 1 & 2 \\ -2 & 1 \end{pmatrix} = \boxed{\frac{-1}{5} \begin{pmatrix} -3 & 4 \\ 4 & 3 \end{pmatrix}}$$

(d) (5 points) Show that $[R_\ell]_\sigma = [\pi_\mathbf{v}]_\sigma - [\pi_\mathbf{w}]_\sigma$.

From parts (b) & (c),

$$[\pi_{\vec{v}}]_\sigma - [\pi_{\vec{\omega}}]_\sigma \overset{(b)}{=} \frac{1}{5} \begin{pmatrix} 1 & 2 \\ 2 & 4 \end{pmatrix} - \frac{1}{5} \begin{pmatrix} 4 & -2 \\ -2 & 1 \end{pmatrix}$$

$$= \frac{1}{5} \begin{pmatrix} -3 & 4 \\ 4 & 3 \end{pmatrix}$$

$$\overset{(c)}{=} [R_\ell]_\sigma \quad \checkmark$$

4. Let

$$A = \begin{pmatrix} 2 & 3 \\ 5 & 4 \end{pmatrix}$$

(a) (5 points) Find the eigenvalues of A.

$$\det(A - \lambda I) = \det\left(\begin{pmatrix} 2 & 3 \\ 5 & 4 \end{pmatrix} - \begin{pmatrix} \lambda & 0 \\ 0 & \lambda \end{pmatrix}\right)$$

$$= \det\begin{pmatrix} 2-\lambda & 3 \\ 5 & 4-\lambda \end{pmatrix}$$

$$= (2-\lambda)(4-\lambda) - 15$$

$$= \lambda^2 - 6\lambda - 7$$

$$= (\lambda - 7)(\lambda + 1)$$

$$= 0$$

$$\Longrightarrow \boxed{\lambda_1 = -1, \ \lambda_2 = 7}$$

(b) (5 points) Find corresponding eigenvectors of A.

$\lambda_1 = -1$: Find $\vec{v} \in N(A - \lambda_1 I) = N\begin{pmatrix} 2-(-1) & 3 \\ 5 & 4-(-1) \end{pmatrix}$

$$= N\begin{pmatrix} 3 & 3 \\ 5 & 5 \end{pmatrix}$$

$$\Longrightarrow \begin{pmatrix} 3 & 3 \\ 5 & 5 \end{pmatrix}\begin{pmatrix} v_1 \\ v_2 \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \end{pmatrix}$$

$\Longrightarrow \begin{pmatrix} 1 & 1 \\ 0 & 0 \end{pmatrix}\begin{pmatrix} v_1 \\ v_2 \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \end{pmatrix}$ after row-reduction

$\Longrightarrow v_1 = -1, v_2 = 1,$ i.e. $\boxed{\vec{v} = \begin{pmatrix} -1 \\ 1 \end{pmatrix}}$

$\lambda_2 = 7 \Longrightarrow$ Need $\vec{\omega} \in N(A - \lambda_2 I) = N\begin{pmatrix} -5 & 3 \\ 5 & -3 \end{pmatrix} \Longrightarrow$

$$\begin{pmatrix} -5 & 3 \\ 5 & -3 \end{pmatrix}\begin{pmatrix} \omega_1 \\ \omega_2 \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \end{pmatrix} \Longrightarrow \begin{pmatrix} 5 & -3 \\ 0 & 0 \end{pmatrix}\begin{pmatrix} \omega_1 \\ \omega_2 \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \end{pmatrix}$$

$\Longrightarrow$ let $\omega_2 = s$, then $\omega_1 = \frac{3}{5}s \Longrightarrow s\begin{pmatrix} 3/5 \\ 1 \end{pmatrix} = \vec{\omega}$

Since I don't like fractions, I'll choose

$s = 5$:

$$\boxed{\vec{\omega} = \underset{\nearrow \text{ plug in 5}}{5}\begin{pmatrix} 3/5 \\ 1 \end{pmatrix} = \begin{pmatrix} 3 \\ 5 \end{pmatrix}}$$

Let's verify our work:

$$A\vec{v} = \begin{pmatrix} 2 & 3 \\ 5 & 4 \end{pmatrix}\begin{pmatrix} -1 \\ 1 \end{pmatrix} = \begin{pmatrix} 1 \\ -1 \end{pmatrix} = (-1)\begin{pmatrix} -1 \\ 1 \end{pmatrix}$$

$$= \lambda_1 \vec{v} \quad \checkmark$$

and

$$A\vec{\omega} = \begin{pmatrix} 2 & 3 \\ 5 & 4 \end{pmatrix}\begin{pmatrix} 3 \\ 5 \end{pmatrix} = \begin{pmatrix} 21 \\ 35 \end{pmatrix} = 7\begin{pmatrix} 3 \\ 5 \end{pmatrix}$$

$$= \lambda_2 \vec{\omega} \quad \checkmark$$

(c) (5 points) Prove directly that the eigenvectors form a basis for $\mathbb{R}^2$.

Let $\beta = (\vec{v}, \vec{w}) = \left(\begin{pmatrix}-1\\1\end{pmatrix}, \begin{pmatrix}3\\5\end{pmatrix}\right)$.

(i) spanning: All $\vec{x} \in \mathbb{R}^2$ lie in span$(\vec{v}, \vec{w}) \iff$

$$\begin{pmatrix}x\\y\end{pmatrix} = a\begin{pmatrix}-1\\1\end{pmatrix} + b\begin{pmatrix}3\\5\end{pmatrix} = \begin{pmatrix}-1 & 3\\1 & 5\end{pmatrix}\begin{pmatrix}a\\b\end{pmatrix} \text{ for some } a, b \in \mathbb{R}$$

But: $\begin{pmatrix}a\\b\end{pmatrix} = \frac{-1}{8}\begin{pmatrix}5 & -3\\-1 & -1\end{pmatrix}\begin{pmatrix}x\\y\end{pmatrix} = \frac{1}{8}\begin{pmatrix}-5 & 3\\1 & 1\end{pmatrix}\begin{pmatrix}x\\y\end{pmatrix}$

will work!

(ii) lin. indp: $a\vec{v} + b\vec{w} = \vec{0} \Rightarrow \begin{pmatrix}\vec{v} & \vec{w}\end{pmatrix}\begin{pmatrix}a\\b\end{pmatrix} = \vec{0}$

$$\Rightarrow \begin{pmatrix}-1 & 3\\1 & 5\end{pmatrix}\begin{pmatrix}a\\b\end{pmatrix} = \begin{pmatrix}0\\0\end{pmatrix}$$

$$\Rightarrow \begin{pmatrix}-1 & 3\\0 & 8\end{pmatrix}\begin{pmatrix}a\\b\end{pmatrix} = \begin{pmatrix}0\\0\end{pmatrix}$$

$$\Rightarrow \boxed{a = b = 0}$$

(d) (5 points) Represent the matrix A in this new basis, that is find $[A]_\beta$.

$$[A]_\beta = M_{\sigma\beta}[A]_\sigma M_{\sigma\beta}^{-1} = M_{\beta\sigma}^{-1} A M_{\beta\sigma}$$

$$= \begin{pmatrix}-1 & 3\\1 & 5\end{pmatrix}^{-1}\begin{pmatrix}2 & 3\\5 & 4\end{pmatrix}\begin{pmatrix}-1 & 3\\1 & 5\end{pmatrix}$$

$$= \frac{-1}{8}\underbrace{\begin{pmatrix}5 & -3\\-1 & -1\end{pmatrix}\begin{pmatrix}2 & 3\\5 & 4\end{pmatrix}}\begin{pmatrix}-1 & 3\\1 & 5\end{pmatrix}$$

$$= -\frac{1}{8}\begin{pmatrix}-5 & 3\\-7 & -7\end{pmatrix}\begin{pmatrix}-1 & 3\\1 & 5\end{pmatrix}$$

$$= \frac{1}{8}\begin{pmatrix}5 & -3\\7 & 7\end{pmatrix}\begin{pmatrix}-1 & 3\\1 & 5\end{pmatrix}$$

$$= \frac{1}{8}\begin{pmatrix}-8 & 0\\0 & 8\cdot 7\end{pmatrix} = \boxed{\begin{pmatrix}-1 & 0\\0 & 7\end{pmatrix}}$$

5. (10 points) Find the volume of the parallelpiped spanned by the vectors

$$\mathbf{u} = \begin{pmatrix} 1 \\ 2 \\ -3 \end{pmatrix}, \quad \mathbf{v} = \begin{pmatrix} 0 \\ -4 \\ -3 \end{pmatrix}, \quad \mathbf{w} = \begin{pmatrix} 7 \\ 2 \\ 0 \end{pmatrix}$$

$$Vol = \left| \det \begin{pmatrix} - & \vec{u} & - \\ - & \vec{v} & - \\ - & \vec{w} & - \end{pmatrix} \right| = \left| \det \begin{pmatrix} | & | & | \\ \vec{u} & \vec{v} & \vec{w} \\ | & | & | \end{pmatrix} \right|$$

$$= \left| \det \begin{pmatrix} 1 & 0 & 7 \\ 2 & -4 & 2 \\ -3 & -3 & 0 \end{pmatrix} \right|$$

$$= \left| 1 \det \begin{pmatrix} -4 & 2 \\ -3 & 0 \end{pmatrix} + 0 + 7 \det \begin{pmatrix} 2 & -4 \\ -3 & -3 \end{pmatrix} \right|$$

$$= \left| 6 + 7(-18) \right|$$

$$= \left| 6 - 126 \right|$$

$$= \left| -120 \right|$$

$$= \boxed{120}$$

6. (5 points) **True or False**: A matrix A is invertible iff 0 is an eigenvalue of A.

$\underline{F}$ If $\lambda = 0$ & $\vec{v} \neq \vec{0}$ is an eigenvector, then $\vec{v} \in N(A - 0I) = N(A) \Rightarrow N(A) \neq \{\vec{0}\}$ $\Rightarrow A$ is $\underline{\text{not}}$ 1-1

7. (5 points) **True or False**: Any matrix A possesses a basis consisting of eigenvalues.

$\underline{F}$ any rotation matrix will work, e.g. $A = R_{\pi/2} = \begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix}$, for $\det(A - \lambda I) = \det\begin{pmatrix} -\lambda & -1 \\ 1 & -\lambda \end{pmatrix} = \lambda^2 + 1 = 0$ has no sol'ns in $\mathbb{R}$.

8. (5 points) **True or False**: Two reflections compose to form a rotation.

$\underline{T}$ proved in HW 6

9. (5 points) **True or False**: A projection $\pi_{\mathbf{v}}$ always has an eigenvalue of 0.

$\underline{T}$, let $\vec{w} = R_{\pi/2}\vec{v}$ then $\pi_v(\vec{w}) = \vec{0} = 0\vec{w}$

$\uparrow_{\lambda = 0}$!

Aagh!

10. (5 points) State the full definition of a vector space V over $\mathbb{R}$.

A vector space V over $\mathbb{R}$ is a set with two operations, $+$ & $\cdot$ (scalar mult.), which satisfy: for all $a, b \in \mathbb{R}$, $u, v, w \in V$,

(1) $u + (v + w) = (u+v) + w$

(2) $u + v = v + u$

(3) $\vec{0} \in V$ exists, satisfying $\vec{0} + v = v = v + \vec{0}$

(4) For all $v \in V$, there is a negative $-v \in V$ satisfying $v + (-v) = 0 = (-v) + v$

(5) $a(bv) = (ab)v$

(6) $(a+b)v = av + bv$

(7) $a(u+v) = au + bv$

(8) $1v = v$ $(1 \in \mathbb{R})$