Linear Algebra Practice Midterm 2
Spring 2019

1. (5 points) State the definition of a basis β for a vector space V.

2. (a) (5 points) Show directly from the definition that the vectors $\mathbf{u} = \begin{pmatrix} 1 \\ 4 \end{pmatrix}$ and $\mathbf{v} = \begin{pmatrix} 5 \\ 6 \end{pmatrix}$ form a basis $\beta = (\mathbf{u}, \mathbf{v})$ for $\mathbb{R}^2$.

   (b) (5 points) Is the basis $\beta = (\mathbf{u}, \mathbf{v})$ positively oriented?

(c) (5 points) Find the β-representation of $\mathbf{x} = \begin{pmatrix} 1 \\ 1 \end{pmatrix}$.

(d) (5 points) Verify that your answer in part (c) is correct: if $[\mathbf{x}]_\beta = \begin{pmatrix} a \\ b \end{pmatrix}$, verify that $\mathbf{x} = a\mathbf{u} + b\mathbf{v}$.

3. Consider the vector $\mathbf{v} = \begin{pmatrix} 1 \\ 2 \end{pmatrix}$ in $\mathbb{R}^2$.

   (a) (5 points) Find the standard representation of the projection $\pi_{\mathbf{v}}$ onto $\mathbf{v}$.

   (b) (5 points) Find the standard representation of the projection $\pi_{\mathbf{w}}$ onto the vector $\mathbf{w} = R_{\pi/2}\mathbf{v}$ orthogonal to $\mathbf{v}$, .

(c) (5 points) Find the standard representation of the reflection R_ℓ across $\ell = \text{span}(\mathbf{v})$.

(d) (5 points) Show that $[R_\ell]_\sigma = [\pi_\mathbf{v}]_\sigma - [\pi_\mathbf{w}]_\sigma$.

4. Let

$$A = \begin{pmatrix} 2 & 3 \\ 5 & 4 \end{pmatrix}$$

(a) (5 points) Find the eigenvalues of A.

(b) (5 points) Find corresponding eigenvectors of A.

(c) (5 points) Prove directly that the eigenvectors form a basis for $\mathbb{R}^2$.

(d) (5 points) Represent the matrix A in this new basis, that is find $[A]_\beta$.

5. (10 points) Find the volume of the parallelpiped spanned by the vectors

$$\mathbf{u} = \begin{pmatrix} 1 \\ 2 \\ -3 \end{pmatrix}, \quad \mathbf{v} = \begin{pmatrix} 0 \\ -4 \\ -3 \end{pmatrix}, \quad \mathbf{w} = \begin{pmatrix} 7 \\ 2 \\ 0 \end{pmatrix}$$

6. (5 points) **True or False**: A matrix A is invertible iff 0 is an eigenvalue of A.

7. (5 points) **True or False**: Any matrix A possesses a basis consisting of eigenvalues.

8. (5 points) **True or False**: Two reflections compose to form a rotation.

9. (5 points) **True or False**: A projection $\pi_{\mathbf{v}}$ always has an eigenvalue of 0.

10. (5 points) State the full definition of a vector space V over $\mathbb{R}$.