Linear Algebra Practice Midterm 1

Spring 2019

1. Let

$$A = \begin{pmatrix} -2 & 3 & -3 & -1 \\ 4 & 1 & 13 & -5 \end{pmatrix}$$

and consider the homogeneous system $A\mathbf{x} = \mathbf{0}$, where $\mathbf{x} \in \mathbb{R}^4$ and $\mathbf{0} \in \mathbb{R}^2$.

(a) Compute $\operatorname{rref}\big(A \mid \mathbf{0}\big)$.

Solution:

$$\operatorname{rref}\big(A \mid \mathbf{0}\big) = \left(\begin{array}{cccc|c} 1 & 0 & 3 & -1 & 0 \\ 0 & 1 & 1 & -1 & 0 \end{array}\right)$$

(b) Identify the pivot columns $\mathbf{b}_j$ in $B = \operatorname{rref}\big(A \mid \mathbf{0}\big)$.

Solution: Let $B = \operatorname{rref}\big(A \mid \mathbf{0}\big)$. Then the pivot columns of B are

$$\mathbf{b}_1 = \begin{pmatrix} 1 \\ 0 \end{pmatrix} \quad \text{and} \quad \mathbf{b}_2 = \begin{pmatrix} 0 \\ 1 \end{pmatrix}$$

(c) Write the remaining, non-pivot, columns of B as linear combinations of the pivot columns you identified in part (b).

Solution: The non-pivot columns (ignoring the last column of zeros) are $\mathbf{b}_3$ and $\mathbf{b}_4$, and

$$\boxed{\begin{aligned}\mathbf{b}_3 &= \begin{pmatrix}3\\1\end{pmatrix}\\ &= 3\begin{pmatrix}1\\0\end{pmatrix} + 1\begin{pmatrix}0\\1\end{pmatrix}\\ &= 3\mathbf{b}_1 + 1\mathbf{b}_2\end{aligned}}$$

and

$$\boxed{\begin{aligned}\mathbf{b}_4 &= \begin{pmatrix}-1\\-1\end{pmatrix}\\ &= (-1)\begin{pmatrix}1\\0\end{pmatrix} + (-1)\begin{pmatrix}0\\1\end{pmatrix}\\ &= (-1)\mathbf{b}_1 + (-1)\mathbf{b}_2\end{aligned}}$$

(d) Returning to the *original* augmented matrix $\left(A \,|\, \mathbf{0}\right)$, show directly that the same dependency relations hold among the columns of A (e.g. if you found that $\mathbf{b}_4 = 5\mathbf{b}_1 + 2\mathbf{b}_3$, then you would demonstrate, starting with the RHS, that $\mathbf{a}_4 = 5\mathbf{a}_1 + 2\mathbf{a}_3$).

Solution: For convenience, let's rewrite A and it's columns here, so we could see what we're doing:

$$A = \begin{pmatrix}-2 & 3 & -3 & -1\\ 4 & 1 & 13 & -5\end{pmatrix},$$

so

$$\mathbf{a}_1 = \begin{pmatrix}-2\\4\end{pmatrix}, \quad \mathbf{a}_2 = \begin{pmatrix}3\\1\end{pmatrix}, \quad \mathbf{a}_3 = \begin{pmatrix}-3\\13\end{pmatrix}, \quad \mathbf{a}_4 = \begin{pmatrix}-1\\-5\end{pmatrix}$$

We start with the RHS of the proposed equation which we're trying to prove, the easier side, and perform the computation, hoping that we get what we're supposed to:

$$\boxed{3\mathbf{a}_1 + 1\mathbf{a}_2 = 3\begin{pmatrix}-2\\4\end{pmatrix} + 1\begin{pmatrix}3\\1\end{pmatrix} = \begin{pmatrix}-6+3\\12+1\end{pmatrix} = \begin{pmatrix}-3\\13\end{pmatrix} = \mathbf{a}_3} \quad \checkmark$$

and similarly

$$\boxed{(-1)\mathbf{a}_1 + (-1)\mathbf{a}_2 = (-1)\begin{pmatrix}-2\\4\end{pmatrix} + (-1)\begin{pmatrix}3\\1\end{pmatrix} = \begin{pmatrix}2-3\\-4-1\end{pmatrix} = \begin{pmatrix}-1\\-5\end{pmatrix} = \mathbf{a}_4} \quad \checkmark$$

(e) Parametrize the free variables, those corresponding to the non-pivot columns of B, and use the form of B to solve for the dependent variables in terms of the independent.

Solution: The pivot columns are 1 and 2, so that means the free variables are x_3 and x_4 in the system $A\mathbf{x} = \mathbf{0}$. Let $\boxed{x_3 = s}$ and $\boxed{x_4 = t}$. We interpret

$$B = \mathrm{rref}\big(A \,|\, \mathbf{0}\big) = \left(\begin{array}{cccc|c} 1 & 0 & 3 & -1 & 0 \\ 0 & 1 & 1 & -1 & 0 \end{array}\right)$$

as

$$\begin{array}{rcl} x_1 \quad + 3x_3 - x_4 & = & 0 \\ x_2 + x_3 - x_4 & = & 0 \end{array}$$

Solving for the dependent variables in terms of the independent gives

$$\boxed{\begin{aligned} x_1 &= -3x_3 + x_4 \\ &= -3s + t \end{aligned} \qquad \text{and} \qquad \begin{aligned} x_2 &= -x_3 + x_4 \\ &= -s + t \end{aligned}}$$

(f) Use the previous part to write *all possible* solutions, namely the null space $N(A)$ of A, as a span of column vectors.

Solution: We simply plug in what we got in part (e): the general solution vector $\mathbf{x} \in \mathbb{R}^4$ of $A\mathbf{x} = \mathbf{0}$ is

$$\boxed{\mathbf{x} = \begin{pmatrix} x_1 \\ x_2 \\ x_3 \\ x_4 \end{pmatrix} = \begin{pmatrix} -3s + t \\ -s + t \\ s \\ t \end{pmatrix} = s \begin{pmatrix} -3 \\ -1 \\ 1 \\ 0 \end{pmatrix} + t \begin{pmatrix} 1 \\ 1 \\ 0 \\ 1 \end{pmatrix}}$$

That is,

$$\boxed{N(A) = \mathrm{span}(\mathbf{u}, \mathbf{v}), \quad \text{where} \;\; \mathbf{u} = \begin{pmatrix} -3 \\ -1 \\ 1 \\ 0 \end{pmatrix}, \quad \mathbf{v} = \begin{pmatrix} 1 \\ 1 \\ 0 \\ 1 \end{pmatrix}}$$

(g) Use the previous step to find two different solutions of $A\mathbf{x} = \mathbf{0}$. Plug them in to demonstrate that they are, in fact, solutions.

Solution: Well, the two vectors $\mathbf{u}$ and $\mathbf{v}$ in part (f) above are two solutions, but so as to get maybe some others, try e.g. letting $s = -2$, $t = 1$ in the expression for the solution vector $\mathbf{x}$, and give it a different name, say

$$\mathbf{w} = (-2)\mathbf{u} + 1\mathbf{v} = (-2)\begin{pmatrix} -3 \\ -1 \\ 1 \\ 0 \end{pmatrix} + 1\begin{pmatrix} 1 \\ 1 \\ 0 \\ 1 \end{pmatrix} = \begin{pmatrix} 7 \\ 3 \\ -2 \\ 1 \end{pmatrix}$$

Let's try these out and make sure they're solution vectors:

$$\begin{aligned} A\mathbf{u} &= \begin{pmatrix} -2 & 3 & -3 & -1 \\ 4 & 1 & 13 & -5 \end{pmatrix}\begin{pmatrix} -3 \\ -1 \\ 1 \\ 0 \end{pmatrix} \\ &= -3\begin{pmatrix} -2 \\ 4 \end{pmatrix} - \begin{pmatrix} 3 \\ 1 \end{pmatrix} + \begin{pmatrix} -3 \\ 13 \end{pmatrix} + 0\begin{pmatrix} -1 \\ -5 \end{pmatrix} \\ &= \begin{pmatrix} 6 - 3 - 3 + 0 \\ -12 - 1 + 13 + 0 \end{pmatrix} \\ &= \begin{pmatrix} 0 \\ 0 \end{pmatrix} \qquad \checkmark \end{aligned}$$

and similarly

$$\begin{aligned} A\mathbf{v} &= \begin{pmatrix} -2 & 3 & -3 & -1 \\ 4 & 1 & 13 & -5 \end{pmatrix}\begin{pmatrix} 1 \\ 1 \\ 0 \\ 1 \end{pmatrix} \\ &= 1\begin{pmatrix} -2 \\ 4 \end{pmatrix} + 1\begin{pmatrix} 3 \\ 1 \end{pmatrix} + 0\begin{pmatrix} -3 \\ 13 \end{pmatrix} + 1\begin{pmatrix} -1 \\ -5 \end{pmatrix} \\ &= \begin{pmatrix} -2 + 3 + 0 - 1 \\ 4 + 1 + 0 - 5 \end{pmatrix} \\ &= \begin{pmatrix} 0 \\ 0 \end{pmatrix} \qquad \checkmark \end{aligned}$$

Lastly,

$$\begin{aligned} A\mathbf{w} &= \begin{pmatrix} -2 & 3 & -3 & -1 \\ 4 & 1 & 13 & -5 \end{pmatrix}\begin{pmatrix} 7 \\ 3 \\ -2 \\ 1 \end{pmatrix} \\ &= 7\begin{pmatrix} -2 \\ 4 \end{pmatrix} + 3\begin{pmatrix} 3 \\ 1 \end{pmatrix} - 2\begin{pmatrix} -3 \\ 13 \end{pmatrix} + 1\begin{pmatrix} -1 \\ -5 \end{pmatrix} \end{aligned}$$

$$= \begin{pmatrix} -14 + 9 + 6 - 1 \\ 28 + 3 - 26 - 5 \end{pmatrix}$$

$$= \begin{pmatrix} 0 \\ 0 \end{pmatrix} \qquad \checkmark$$

2. Let $\mathbf{u} = \begin{pmatrix} 1 \\ 2 \\ -3 \end{pmatrix}$, $\mathbf{v} = \begin{pmatrix} 0 \\ -4 \\ -3 \end{pmatrix}$, $\mathbf{w} = \begin{pmatrix} 7 \\ 2 \\ 0 \end{pmatrix} \in \mathbb{R}^3$. Are these three vectors linearly independent? If not, write $\mathbf{w}$ as a linear combination of $\mathbf{u}$ and $\mathbf{v}$.

Solution: The definition of linear independence, we recall, says "If $a\mathbf{u}+b\mathbf{v}+c\mathbf{w} = \mathbf{0}$, then $a = b = c = 0$." Let's see whether this is the case for our $\mathbf{u}$, $\mathbf{v}$ and $\mathbf{w}$:

Suppose there are scalars $a, b, c \in \mathbb{R}$ satisfying $a\mathbf{u} + b\mathbf{v} + c\mathbf{w} = \mathbf{0}$. Because in the case of vectors in $\mathbb{R}^n$ this amounts to a matrix product (*by the* *definition matrix product!*), we can *rephrase* the original problem in terms of a system, then use row-reduction:

$$\begin{aligned} \mathbf{0} &= a\mathbf{u} + b\mathbf{v} + c\mathbf{w} \\ &= a\begin{pmatrix} 1 \\ 2 \\ -3 \end{pmatrix} + b\begin{pmatrix} 0 \\ -4 \\ -3 \end{pmatrix} + c\begin{pmatrix} 7 \\ 2 \\ 0 \end{pmatrix} \\ &= \begin{pmatrix} 1 & 0 & 7 \\ 2 & -4 & 2 \\ -3 & -3 & 0 \end{pmatrix}\begin{pmatrix} a \\ b \\ c \end{pmatrix} \\ &= A\mathbf{x} \end{aligned}$$

where

$$A = \begin{pmatrix} 1 & 0 & 7 \\ 2 & -4 & 2 \\ -3 & -3 & 0 \end{pmatrix}, \qquad \mathbf{x} = \begin{pmatrix} a \\ b \\ c \end{pmatrix}$$

Let's see whether $\mathbf{x} = 0$ or not, then, using the matrix method of row-reduction on $(A \mid \mathbf{0})$: since

$$\operatorname{rref}(A \mid \mathbf{0}) = \left(\begin{array}{ccc|c} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \end{array}\right)$$

we conclude that $a = b = c = 0$, or $\mathbf{x} = 0$. Thus,

$$\boxed{\mathbf{u}, \mathbf{v} \text{ and } \mathbf{w} \text{ are linearly independent!}}$$

3. **True or False**: If a matrix A has trivial null space, $N(A) = \{\mathbf{0}\}$, then as a function A is one-to-one.

   Solution: $\boxed{\text{True.}}$ To see this, suppose there were two vectors $\mathbf{x}_1, \mathbf{x}_2 \in \mathbb{R}^n$ satisfying $A\mathbf{x}_1 = A\mathbf{x}_2$ (i.e. $\mathbf{x}_1$ and $\mathbf{x}_2$ are sent to the same $\mathbf{y}$-value). Then, subtracting $A\mathbf{x}_2$ from both sides and factoring A out gives

   $$\mathbf{0} = A\mathbf{x}_1 - A\mathbf{x}_2 = A(\mathbf{x}_1 - \mathbf{x}_2)$$

   But this means $\mathbf{x}_1 - \mathbf{x}_2 \in N(A) = \{\mathbf{0}\}$, and therefore

   $$\mathbf{x}_1 - \mathbf{x}_2 = \mathbf{0}$$

   from which it follows that in fact $\mathbf{x}_1 = \mathbf{x}_2$. That is, there can only be one $\mathbf{x} \in \mathbb{R}^n$ to each $\mathbf{y} = A\mathbf{x} \in \mathbb{R}^m$. A is one-to-one.

4. **True or False**: If $A \in M_{3,5}(\mathbb{R})$, then as a function A cannot be one-to-one.

   Solution: $\boxed{\text{True.}}$ Actually, A *must* have nontrivial null space $N(A)$, because it can have max 3 pivot columns, therefore *at least* 2 free variables. Parametrizing them as in number 1 above would give *two linearly independent* solution vectors of $A\mathbf{x} = \mathbf{0}$, nevermind all their possible linear combinations, which would also be solutions. Thus, there are in fact infinitely many vectors sent to the same $\mathbf{y}$-value of $\mathbf{0}$, and this merely by dimension-count, because $3 < 5$.

5. **True or False**: The matrix A in problem #1 above is onto.

   Solution: $\boxed{\text{True.}}$ First, A can be thought of as a function $\mathbb{R}^4 \to \mathbb{R}^2$, and the range $R(A)$ is a subspace of its target space $\mathbb{R}^2$, which moreover, *by the definition of matrix multiplication*, is characterized by

   $$R(A) = \text{span}\big(\mathbf{a}_1, \mathbf{a}_2, \mathbf{a}_3, \mathbf{a}_4\big)$$

   Since two of the columns are pivots, namely $\mathbf{a}_1 = \begin{pmatrix} -2 \\ 4 \end{pmatrix}$, $\mathbf{a}_2 = \begin{pmatrix} 3 \\ 1 \end{pmatrix}$, and those are always linearly independent (since we have seen that the dependency relations among the columns of $B = \text{rref}(A)$ are the same as those among the columns of A, and this is true of $\mathbf{b}_1$ and $\mathbf{b}_2$), this reduces the span to

   $$R(A) = \text{span}(\mathbf{a}_1, \mathbf{a}_2)$$

   (since $\mathbf{a}_3, \mathbf{a}_4 \in \text{span}(\mathbf{a}_1, \mathbf{a}_2)$ already, as we showed in # 1(d)). The matrix $C \in M_2(\mathbb{R})$ whose columns are $\mathbf{a}_1$ and $\mathbf{a}_2$, as a function $\mathbb{R}^2 \to \mathbb{R}^2$, is one-to-one (because it's columns are linearly independent) and onto (since if $\mathbf{y} \in \mathbb{R}^2$, then

   $$\text{rref}\big(\mathbf{a}_1 \ \mathbf{a}_2 \mid \mathbf{y}\big) = \left(\begin{array}{cc|c} 1 & 0 & h \\ 0 & 1 & k \end{array}\right)$$

   for some $h, k \in \mathbb{R}$, namely the components of the solution $\mathbf{x}$ in $C\mathbf{x} = \mathbf{y}$, whence $R(C) = \mathbb{R}^2$ and A is onto). C is therefore invertible. Alternatively, we can compute the inverse directly,

   $$C^{-1} = \begin{pmatrix} -2 & 3 \\ 4 & 1 \end{pmatrix}^{-1} = \frac{1}{(-2)\cdot 1 - 4 \cdot 3} \begin{pmatrix} 1 & -3 \\ -4 & -2 \end{pmatrix} = \begin{pmatrix} -1/16 & 3/16 \\ 1/4 & 1/8 \end{pmatrix}$$

   Either way, we can see that C is onto, and since C has range the same as A, A is onto.

6. **True or False**: A square matrix A is invertible (i.e. both one-to-one and onto as a function) if and only if the columns of A are linearly independent.

Solution: True. Linear indepndence in $\mathbb{R}^n$ can be rephrased in terms of a matrix equation (by the definition of matrix multiplication), so the statement "A is linearly independent" is, by definition, equivalent to the statement "If $\sum_{i=1}^n c_i \mathbf{a}_i = \mathbf{0}$, then all $c_i = 0$," but by the definition of matrix multiplication this statement is equivalent to "If $A\mathbf{x} = \mathbf{0}$, where $\mathbf{x}$ is the column vector with components the c_i, then $\mathbf{x} = \mathbf{0}$." Thus, to say all $c_i = 0$ and to say the column vector $\mathbf{x} = \mathbf{0}$ are equivalent statements.

What does this show? It shows that "The columns of A are linearly independent" iff "The null space of A is trivial," i.e. $N(A) = \{\mathbf{0}\}$, and this is equivalent to "A is *one-to-one*" by #3 above. What about *onto*? This is where the *square* part of A comes in. We know that A is $n \times n$, so if it has n linearly independent columns, the system $A\mathbf{x} = \mathbf{y}$ will have a unique solution for each $\mathbf{y} \in \mathbb{R}^n$, as $\text{rref}\big(A \,|\, \mathbf{y}\big)$ will be $\big(I \,|\, \mathbf{h}\big)$, which means $\mathbf{x} = \mathbf{h}$ is a solution. Thus, $R(A) = \mathbb{R}^n$ and A is onto.

Remark 0.1 Observe that the *uniqueness* of $\mathbf{x}$ actually is the one-to-one part. That part of the problem could be attacked in this row-reduction way, but the connection to linear independence above still needs to be made via the null space $N(A)$, as in number 3. ■

7. Let $A = \begin{pmatrix} 1 & 2 \\ 3 & 4 \end{pmatrix} \in M_2(\mathbb{R})$ and $B = \begin{pmatrix} -1 & 2 & -3 \\ 4 & -5 & 6 \end{pmatrix} \in M_{2,3}(\mathbb{R})$. Compute $A^{-1}B$.

Solution: $A^{-1} = \dfrac{1}{1 \cdot 4 - 3 \cdot 2} \begin{pmatrix} 4 & -2 \\ -3 & 1 \end{pmatrix} = -\dfrac{1}{2} \begin{pmatrix} 4 & -2 \\ -3 & 1 \end{pmatrix}$, which I'll leave unscaled for now, to make the computation easier. Then, using the rows of A^{-1} (prior to scaling by $-1/2$) and the columns of B,

$$\begin{aligned}
A^{-1}B &= -\frac{1}{2} \begin{pmatrix} 4 & -2 \\ -3 & 1 \end{pmatrix} \begin{pmatrix} -1 & 2 & -3 \\ 4 & -5 & 6 \end{pmatrix} \\
&= -\frac{1}{2} \begin{pmatrix} \vec{A}_1^{-1} \cdot \mathbf{b}_1 & \vec{A}_1^{-1} \cdot \mathbf{b}_2 & \vec{A}_1^{-1} \cdot \mathbf{b}_3 \\ \vec{A}_2^{-1} \cdot \mathbf{b}_1 & \vec{A}_2^{-1} \cdot \mathbf{b}_2 & \vec{A}_2^{-1} \cdot \mathbf{b}_3 \end{pmatrix} \\
&= -\frac{1}{2} \begin{pmatrix} \langle 4, -2 \rangle \cdot \binom{-1}{4} & \langle 4, -2 \rangle \cdot \binom{2}{-5} & \langle 4, -2 \rangle \cdot \binom{-3}{6} \\ \langle -3, 1 \rangle \cdot \binom{-1}{4} & \langle -3, 1 \rangle \cdot \binom{2}{-5} & \langle -3, 1 \rangle \cdot \binom{-3}{6} \end{pmatrix} \\
&= -\frac{1}{2} \begin{pmatrix} -12 & 18 & -24 \\ 7 & -11 & 15 \end{pmatrix} \\
&= \boxed{\begin{pmatrix} 6 & -9 & 12 \\ -\frac{7}{2} & \frac{11}{2} & -\frac{15}{2} \end{pmatrix}}
\end{aligned}$$