Solutions to Linear Algebra Practice Midterm
Summer 2011

1. (a) Let $x_2 = t$, then $x_1 = -2t$. Since $x_3 = x_4 = x_5 = 0$, all solutions have the form $\mathbf{x} = (-2t, t, 0, 0, 0) = t(-2, 1, 0, 0, 0)$, so

$$\boxed{\ker A = \operatorname{span}\left(\begin{bmatrix} -2 \\ 1 \\ 0 \\ 0 \\ 0 \end{bmatrix}\right)}$$

(b) By looking at rref(A) we know that $\mathbf{a}_1, \mathbf{a}_3, \mathbf{a}_4, \mathbf{a}_5$ are linearly independent and form a basis for im A, so

$$\boxed{\operatorname{im} A = \operatorname{span}(\mathbf{a}_1, \mathbf{a}_3, \mathbf{a}_4, \mathbf{a}_5) = \operatorname{span}\left(\begin{bmatrix} 1 \\ 4 \\ 3 \\ 2 \end{bmatrix}, \begin{bmatrix} 3 \\ 1 \\ 1 \\ 1 \end{bmatrix}, \begin{bmatrix} 2 \\ 1 \\ 2 \\ 9 \end{bmatrix}, \begin{bmatrix} 0 \\ 6 \\ 5 \\ 10 \end{bmatrix}\right) = \mathbb{R}^4}$$

(c) Looking at rref$([A|\mathbf{b}])$ we see right away that $x_3 = -18$, $x_4 = 16$ and $x_5 = -16$. Also, $x_1 + 2x_2 = 27$. Pick x_1, say $x_1 = 1$, then $x_2 = 13$, and one solution, then, is

$$\boxed{\mathbf{s} = \begin{bmatrix} 1 \\ 13 \\ -18 \\ 16 \\ -16 \end{bmatrix}}$$

(d) We know that the solution set K is of the form $K = \mathbf{s} + \ker A$, so

$$\boxed{K = \begin{bmatrix} 1 \\ 13 \\ -18 \\ 16 \\ -16 \end{bmatrix} + \operatorname{span}\left(\begin{bmatrix} -2 \\ 1 \\ 0 \\ 0 \\ 0 \end{bmatrix}\right) = \left\{ \begin{bmatrix} 1 \\ 13 \\ -18 \\ 16 \\ -16 \end{bmatrix} + t \begin{bmatrix} -2 \\ 1 \\ 0 \\ 0 \\ 0 \end{bmatrix} \;\middle|\; t \in \mathbb{R} \right\}}$$

(e) $\boxed{\mathbf{s}_2 = \begin{bmatrix} 1 \\ 13 \\ -18 \\ 16 \\ -16 \end{bmatrix} + 3 \begin{bmatrix} -2 \\ 1 \\ 0 \\ 0 \\ 0 \end{bmatrix} = \begin{bmatrix} -5 \\ 16 \\ -18 \\ 16 \\ -16 \end{bmatrix}}$. Indeed,

$$A\mathbf{s}_2 = \begin{bmatrix} 1 & 2 & 3 & 2 & 0 \\ 4 & 8 & 1 & 1 & 6 \\ 3 & 6 & 1 & 2 & 5 \\ 2 & 4 & 1 & 9 & 10 \end{bmatrix} \begin{bmatrix} -5 \\ 16 \\ -18 \\ 16 \\ -16 \end{bmatrix} = \begin{bmatrix} -5 + 32 - 54 + 32 \\ -20 + 128 - 18 + 16 - 96 \\ -15 + 96 - 18 + 32 - 80 \\ -10 + 64 - 18 + 144 - 160 \end{bmatrix} = \begin{bmatrix} 5 \\ 10 \\ 15 \\ 20 \end{bmatrix}$$

2. Note, this is only true for $n \geq 2$. Suppose $A \in \mathbb{R}^{n \times n}$, $n \geq 2$, were invertible, so that A^{-1} existed. Then

$$A = AI_n = A(AA^{-1}) = (AA)A^{-1} = A^2A^{-1} = AA^{-1} = I_n$$

But I_n has zero entries if $n \geq 2$.

3. This is not always true for matrices, because AB does not necessarily equal BA, and so $(A+B)^2 = A^2 + AB + BA + B^2$ does not always equal $A^2 + 2AB + B^2$. For example, if $A = \begin{bmatrix} 1 & 2 \\ 3 & 4 \end{bmatrix}$ and $B = \begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix}$, then

$$(A+B)^2 = \left(\begin{bmatrix} 1 & 2 \\ 3 & 4 \end{bmatrix} + \begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix} \right)^2 = \begin{bmatrix} 1 & 3 \\ 4 & 4 \end{bmatrix}^2 = \begin{bmatrix} 13 & 15 \\ 20 & 28 \end{bmatrix}$$

while

$$\begin{aligned} A^2 + 2AB + B^2 &= \begin{bmatrix} 1 & 2 \\ 3 & 4 \end{bmatrix}^2 + 2 \begin{bmatrix} 1 & 2 \\ 3 & 4 \end{bmatrix} \begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix} + \begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix}^2 \\ &= \begin{bmatrix} 7 & 10 \\ 15 & 22 \end{bmatrix} + \begin{bmatrix} 4 & 2 \\ 8 & 6 \end{bmatrix} + \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix} \\ &= \begin{bmatrix} 12 & 12 \\ 23 & 29 \end{bmatrix} \end{aligned}$$

4. Let $A = \begin{bmatrix} 2 & 4 & 8 \\ 4 & 5 & 1 \\ 7 & 9 & 3 \end{bmatrix}$.

   (a) Let's row-reduce $[A|\mathbf{b}]$, with $\mathbf{b}$ from part (b), because then we'll get $\operatorname{rref}(A)$ and $\operatorname{rref}([A|\mathbf{b}])$ in one go. After some work, we get

   $$\operatorname{rref}([A|\mathbf{b}]) = \begin{bmatrix} 1 & 0 & -6 & 1 \\ 0 & 1 & 5 & 1 \\ 0 & 0 & 0 & 0 \end{bmatrix}$$

   From just $\operatorname{rref}(A)$ we get, by letting $z = t$, that $x = 6t$ and $y = -5t$, which implies

   $$\boxed{\ker A = \left\{ t \begin{bmatrix} 6 \\ -5 \\ 1 \end{bmatrix} \;\middle|\; t \in \mathbb{R} \right\} = \operatorname{span}\left(\begin{bmatrix} 6 \\ -5 \\ 1 \end{bmatrix} \right)}$$

   The image of A is easy, we just read off the linearly independent columns of A from those of $\operatorname{rref}(A)$, namely the first two. Thus,

   $$\boxed{\operatorname{im} A = \operatorname{span}(\mathbf{a}_1, \mathbf{a}_2) = \operatorname{span}\left(\begin{bmatrix} 2 \\ 4 \\ 7 \end{bmatrix}, \begin{bmatrix} 4 \\ 5 \\ 9 \end{bmatrix} \right)}$$

   (b) To get all solutions we merely need to find one, then add it to $\ker A$. Well, from $\operatorname{rref}([A|\mathbf{b}])$ we get that

   $$\begin{aligned} x - 6z &= 1 \\ y + 5z &= 1 \end{aligned}$$

   Let's make it as easy as possible, let $z = 0$. Then $x = y = 1$, and one solution is $\mathbf{s} = (1, 1, 0)$, and so the entire solution set is

   $$\boxed{K = \begin{bmatrix} 1 \\ 1 \\ 0 \end{bmatrix} + \ker A = \begin{bmatrix} 1 \\ 1 \\ 0 \end{bmatrix} + \operatorname{span}\left(\begin{bmatrix} 6 \\ -5 \\ 1 \end{bmatrix} \right)}$$

5. Let B be the matrix whose columns are the basis vectors in β, $B = \begin{bmatrix} 1 & 1 \\ 1 & 2 \end{bmatrix}$. We know that $\mathbf{x} = B[\mathbf{x}]_\beta$, from which it follows that $[\mathbf{x}]_\beta = B^{-1}\mathbf{x}$. B^{-1} is easy to find, just use the formula $A^{-1} = \frac{1}{ad-bc}\begin{bmatrix} d & -b \\ -c & a \end{bmatrix}$. Then,

$$\boxed{[\mathbf{x}]_\beta = B^{-1}\mathbf{x} = \begin{bmatrix} 2 & -1 \\ -1 & 1 \end{bmatrix}\begin{bmatrix} -2 \\ 5 \end{bmatrix} = \begin{bmatrix} -9 \\ 7 \end{bmatrix}}$$

It's easy to check that indeed $\begin{bmatrix} -2 \\ 5 \end{bmatrix} = -9\begin{bmatrix} 1 \\ 1 \end{bmatrix} + 7\begin{bmatrix} 1 \\ 2 \end{bmatrix}$.

6. Let B be the matrix whose columns are the basis vectors in β, $B = \begin{bmatrix} 1 & 1 \\ 1 & 0 \end{bmatrix}$. Then $B^{-1} = \begin{bmatrix} 0 & 1 \\ 1 & -1 \end{bmatrix}$, so that

$$[T_A(\mathbf{e}_1)]_\beta = \frac{1}{\sqrt{2}}\begin{bmatrix} 1 \\ 1 \end{bmatrix}_\beta = B^{-1}\frac{1}{\sqrt{2}}\begin{bmatrix} 1 \\ 1 \end{bmatrix} = \frac{1}{\sqrt{2}}\begin{bmatrix} 0 & 1 \\ 1 & -1 \end{bmatrix}\begin{bmatrix} 1 \\ 1 \end{bmatrix} = \begin{bmatrix} 1/\sqrt{2} \\ 0 \end{bmatrix}$$

Similarly,

$$[T_A(\mathbf{e}_2)]_\beta = \begin{bmatrix} 1/\sqrt{2} \\ -\sqrt{2} \end{bmatrix}$$

and therefore

$$\boxed{[T_A]_{\rho,\beta} = \begin{bmatrix} [T_A(\mathbf{e}_1)]_\beta & [T_A(\mathbf{e}_2)]_\beta \end{bmatrix} = \begin{bmatrix} 1/\sqrt{2} & 1/\sqrt{2} \\ 0 & -\sqrt{2} \end{bmatrix}}$$