Linear Algebra Midterm
Summer 2011

1. Consider the system of equations

$$\begin{aligned} x_1 + \ 2x_2 + \ 3x_3 + \ 2x_4 \qquad\quad &= 5 \\ 4x_1 + \ 8x_2 + \ x_3 + \ x_4 + \ 6x_5 &= 10 \\ 3x_1 + \ 6x_2 + \ x_3 + \ 2x_4 + \ 5x_5 &= 15 \\ 2x_1 + \ 4x_2 + \ x_3 + \ 9x_4 + \ 10x_5 &= 30 \end{aligned}$$

This can be succintly written as $A\mathbf{x} = \mathbf{b}$ and represented in matrix form as $[A|\mathbf{b}] \in \mathbb{R}^{4\times 6}$. Given that

$$\text{rref}([A|\mathbf{b}]) = \begin{bmatrix} 1 & 2 & 0 & 0 & 0 & 27 \\ 0 & 0 & 1 & 0 & 0 & -18 \\ 0 & 0 & 0 & 1 & 0 & 16 \\ 0 & 0 & 0 & 0 & 1 & -16 \end{bmatrix}$$

(a) Find $\ker A$ as a span of vectors. (You'll need to consider $\text{rref}([A|\mathbf{0}])$ here.)

(b) Find $\text{im}\, A$ as a span of vectors.

(c) Find a particular solution $\mathbf{s}$ of the system $A\mathbf{x} = \mathbf{b}$.

(d) Use parts (a) and (c) to find the set of all possible solutions $K = \mathbf{s} + \ker A$, and write this in the form $K = \{\mathbf{s} + t_1\mathbf{v}_1 + \cdots + t_k\mathbf{v}_k \mid t_1, \ldots, t_k \in \mathbb{R}\}$ where the vectors $\mathbf{v}_i$ are the basis vectors of $\ker A$.

(e) Use part (d) with $t_1 = \cdots = t_k = 3$ to find a second solution $\mathbf{s}_2$ of $A\mathbf{x} = \mathbf{b}$ and verify that this is indeed a solution.

2. Let $A \in \mathbb{R}^{n\times n}$ satisfy $A^2 = A$. Show that if all entries of A are nonzero, then A is not invertible. [Hint: It's easier to prove the contrapositive of this statement.]

3. If a and b are real numbers, we know that $(a+b)^2 = a^2 + 2ab + b^2$. If $A, B \in \mathbb{R}^{n \times n}$, is the equation $(A+B)^2 = A^2 + 2AB + B^2$ still true? If so, prove it, if not, find examples of matrices A and B for which this fails.

4. Let $A = \begin{bmatrix} 2 & 4 & 8 \\ 4 & 5 & 1 \\ 7 & 9 & 3 \end{bmatrix}$.

   (a) Find $\ker A$ and $\operatorname{im} A$ as spans of vectors.

(b) Find all solutions of the system $A\mathbf{x} = \mathbf{b}$, where $\mathbf{b} = \begin{bmatrix} 6 \\ 9 \\ 16 \end{bmatrix}$.

5. Consider the basis $\beta = \left\{ \begin{bmatrix} 1 \\ 1 \end{bmatrix}, \begin{bmatrix} 1 \\ 2 \end{bmatrix} \right\}$ for $\mathbb{R}^2$. Find the representation $[\mathbf{x}]_\beta$ of the vector $\mathbf{x} = \begin{bmatrix} -2 \\ 5 \end{bmatrix}$ in this basis.

6. Let $\rho = \{\mathbf{e}_1, \mathbf{e}_2\}$ be the standard basis for $\mathbb{R}^2$ and let $\beta = \left\{ \begin{bmatrix} 1 \\ 1 \end{bmatrix}, \begin{bmatrix} 1 \\ 0 \end{bmatrix} \right\}$ be another basis for $\mathbb{R}^2$. If $T_A \in \mathcal{L}(\mathbb{R}^2, \mathbb{R}^2)$ is the counterclockwise rotation through $\pi/4$, with associated matrix

$$A = \begin{bmatrix} \frac{1}{\sqrt{2}} & -\frac{1}{\sqrt{2}} \\ \frac{1}{\sqrt{2}} & \frac{1}{\sqrt{2}} \end{bmatrix}$$

find the matrix representation $[T_A]_{\rho,\beta}$ of T_A in these bases.