Linear Algebra Practice Final
Summer 2011

1. Let

$$A = \begin{bmatrix} 3 & -1 & 3 & -1 & 2 \\ 1 & -1 & -1 & -2 & -1 \\ 5 & -2 & 1 & -3 & 3 \\ 2 & -1 & 0 & -2 & 1 \end{bmatrix}$$

   (a) First,

$$\text{rref}([A|\mathbf{0}]) = \begin{bmatrix} 1 & 0 & 0 & 0 & 2 & 0 \\ 0 & 1 & 0 & 0 & 5 & 0 \\ 0 & 0 & 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 & -1 & 0 \end{bmatrix}$$

   so if we let $x_5 = t$, then $x_4 = t$, $x_3 = 0$, $x_2 = -5t$, and $x_1 = -2t$, so that

$$\mathbf{x} = \begin{bmatrix} -2t \\ -5t \\ 0 \\ t \\ t \end{bmatrix} = t \begin{bmatrix} -2 \\ -5 \\ 0 \\ 1 \\ 1 \end{bmatrix}$$

   whence $\ker A = \text{span}(-2, -5, 0, 1, 1)$ and a basis for $\ker A$ is

$$\boxed{\beta = \left\{ \begin{bmatrix} -2 \\ -5 \\ 0 \\ 1 \\ 1 \end{bmatrix} \right\}}$$

   (b) To find a particular solution, $\mathbf{s}$, we consider

$$\text{rref}([A|\mathbf{b}]) = \text{rref} \begin{bmatrix} 3 & -1 & 3 & -1 & 2 & 5 \\ 1 & -1 & -1 & -2 & -1 & 2 \\ 5 & -2 & 1 & -3 & 3 & 10 \\ 2 & -1 & 0 & -2 & 1 & 5 \end{bmatrix} = \begin{bmatrix} 1 & 0 & 0 & 0 & 2 & 3 \\ 0 & 1 & 0 & 0 & 5 & 7 \\ 0 & 0 & 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 & -1 & -3 \end{bmatrix}$$

   Make it easy on yourself and let $x_5 = 0$. Then $x_1 = 3$, $x_2 = 7$, $x_3 = 0$ and $x_4 = -3$, so one solution is this one:

$$\mathbf{s} = \begin{bmatrix} 3 \\ 7 \\ 0 \\ -3 \\ 0 \end{bmatrix}$$

   Consequently,

$$\boxed{K = \mathbf{s} + \ker A = \left\{ \begin{bmatrix} 3 \\ 7 \\ 0 \\ -3 \\ 0 \end{bmatrix} + t \begin{bmatrix} -2 \\ -5 \\ 0 \\ 1 \\ 1 \end{bmatrix} \;\middle|\; t \in \mathbb{R} \right\}}$$

(c) Well, one solution is $\mathbf{s}$ above, and another may be gotten by picking $t = 1$, say, to get

$$\mathbf{s}_2 = \begin{bmatrix} 3 \\ 7 \\ 0 \\ -3 \\ 0 \end{bmatrix} + \begin{bmatrix} -2 \\ -5 \\ 0 \\ 1 \\ 1 \end{bmatrix} = \begin{bmatrix} 1 \\ 2 \\ 0 \\ -2 \\ 1 \end{bmatrix}$$

It's easy to verify that these are indeed solutions of the original system.

2. We know that S is too large, because it contains 5 elements while $\mathbb{R}_3[x]$ is a 4-dimensional space. So we will have to reduce it, the question is just by how much. Well, it's easier to deal with matrices, so why don't we represent the vectors in S in the standard basis $\rho = \{1, x, x^2, x^3\}$ for $\mathbb{R}_3[x]$, and put the resulting column vectors in a matrix:

$$\begin{aligned} A &= \begin{bmatrix} [2+x+x^2+3x^3]_\rho & [4+2x+4x^2+6x^3]_\rho & [6+3x+8x^2+7x^3]_\rho & [2+x+5x^3]_\rho & [4+x+9x^3]_\rho \end{bmatrix} \\ &= \begin{bmatrix} 2 & 4 & 6 & 2 & 4 \\ 1 & 2 & 3 & 1 & 1 \\ 1 & 4 & 8 & 0 & 0 \\ 3 & 6 & 7 & 5 & 9 \end{bmatrix} \end{aligned}$$

Row-reducing A gives

$$\text{rref}(A) = \begin{bmatrix} 1 & 0 & 0 & 0 & 0 \\ 0 & 1 & 0 & 2 & 0 \\ 0 & 0 & 1 & -1 & 0 \\ 0 & 0 & 0 & 0 & 1 \end{bmatrix}$$

This tells us that the first three columns and the last are linearly independent, so that the first three and the last polynomials in S are linearly independent. These will form a basis for span S, since they are linearly independent and span S:

$$\boxed{\beta = \{2+x+x^2+3x^3,\ 4+2x+4x^2+6x^3,\ 6+3x+8x^2+7x^3,\ 4+x+9x^3\}}$$

3. The problem with A is that $c_A(x) = \det(A - xI_2) = \det \begin{bmatrix} -x & 1 \\ 0 & -x \end{bmatrix} = x^2$, so $\lambda = 0$ is the only eigenvalue, and it has algebraic multiplicity 2, whereas

$$\dim(E_0) = \dim(\ker(A - 0I)) = \dim(\ker(A)) = 1$$

The problem with B is that $c_B(x) = \det(B - xI_2) = \det \begin{bmatrix} -x & 1 \\ -1 & -x \end{bmatrix} = x^2 + 1$ doesn't factor. That alone is enough, but you'll also note that B consequently doesn't even have eigenvalues, and so doesn't have eigenvectors, which are needed in any basis that will diagonalize it.

4. Let

$$A = \begin{bmatrix} 3 & 1 & 1 \\ 2 & 4 & 2 \\ -1 & -1 & 1 \end{bmatrix}$$

Then

$$\begin{aligned} c_A(x) &= \det(A - xI_3) \\ &= \det \begin{bmatrix} 3-x & 1 & 1 \\ 2 & 4-x & 2 \\ -1 & -1 & 1-x \end{bmatrix} \\ &= -x^3 + 8x^2 - 20x + 16 \\ &= (-1)(x-4)(x-2)^2 \end{aligned}$$

so the eigenvalues of A are $\lambda_1 = 4$ and $\lambda_2 = 2$, with algebraic multiplicities 1 and 2, respectively. Let's find the dimensions of E_4 and E_2: first,

$$E_4 = \ker(A - 4I) = \ker \begin{bmatrix} -1 & 1 & 1 \\ 2 & 0 & 2 \\ -1 & -1 & -3 \end{bmatrix} = \ker \overbrace{\begin{bmatrix} 1 & 0 & 1 \\ 0 & 1 & 2 \\ 0 & 0 & 0 \end{bmatrix}}^{=\operatorname{rref}(A-4I)}$$

has dimension 1, because the rank of this matrix is 2, and

$$E_2 = \ker(A - 2I) = \ker \begin{bmatrix} 1 & 1 & 1 \\ 2 & 2 & 2 \\ -1 & -1 & -1 \end{bmatrix} = \ker \overbrace{\begin{bmatrix} 1 & 0 & 1 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix}}^{=\operatorname{rref}(A-2I)}$$

has dimension 2, because the rank of this matrix is 1. Consequently, the dimensions of the eigenspaces equal the algebraic multiplicities of the corresponding eigenvalues, so A is diagonalizable.

To find a basis β for V that diagonalizes A, we merely have to find bases for E_4 and E_2. This is easy once we look at $\operatorname{rref}(A - 4I)$ and $\operatorname{rref}(A - 2I)$. In the first case, letting $z = t$, we get that $x = -t$ and $y = -2t$, so that

$$E_4 = \operatorname{span}\left(\begin{bmatrix} -1 \\ -2 \\ 1 \end{bmatrix} \right)$$

and similarly with E_2, letting $y = s$ and $z = t$ gives that $x = -s - t$, so that

$$\begin{bmatrix} x \\ y \\ z \end{bmatrix} = \begin{bmatrix} -s - t \\ s \\ t \end{bmatrix} = s \begin{bmatrix} -1 \\ 1 \\ 0 \end{bmatrix} + t \begin{bmatrix} -1 \\ 0 \\ 1 \end{bmatrix}$$

whence

$$E_2 = \operatorname{span}\left(\begin{bmatrix} -1 \\ 1 \\ 0 \end{bmatrix}, \begin{bmatrix} -1 \\ 0 \\ 1 \end{bmatrix} \right)$$

The vectors in these spans are linearly independent, and thus form bases for E_4 and E_2, respectively. Joining all three into one basis gives a diagonalizing basis

$$\boxed{\beta = \left\{ \begin{bmatrix} -1 \\ -2 \\ 1 \end{bmatrix}, \begin{bmatrix} -1 \\ 1 \\ 0 \end{bmatrix}, \begin{bmatrix} -1 \\ 0 \\ 1 \end{bmatrix} \right\}}$$

To see that this is true, we compute:

$$T_A(\mathbf{b}_1) = A\mathbf{b}_1 = \begin{bmatrix} 3 & 1 & 1 \\ 2 & 4 & 2 \\ -1 & -1 & 1 \end{bmatrix} \begin{bmatrix} -1 \\ -2 \\ 1 \end{bmatrix} = \begin{bmatrix} -4 \\ -8 \\ 4 \end{bmatrix} = 4 \begin{bmatrix} -1 \\ -2 \\ 1 \end{bmatrix} = 4\mathbf{b}_1$$

$$T_A(\mathbf{b}_2) = A\mathbf{b}_2 = \begin{bmatrix} 3 & 1 & 1 \\ 2 & 4 & 2 \\ -1 & -1 & 1 \end{bmatrix} \begin{bmatrix} -1 \\ 1 \\ 0 \end{bmatrix} = \begin{bmatrix} -2 \\ 2 \\ 0 \end{bmatrix} = 2 \begin{bmatrix} -1 \\ 1 \\ 0 \end{bmatrix} = 2\mathbf{b}_2$$

$$T_A(\mathbf{b}_3) = A\mathbf{b}_3 = \begin{bmatrix} 3 & 1 & 1 \\ 2 & 4 & 2 \\ -1 & -1 & 1 \end{bmatrix} \begin{bmatrix} -1 \\ 0 \\ 1 \end{bmatrix} = \begin{bmatrix} -2 \\ 0 \\ 2 \end{bmatrix} = 2 \begin{bmatrix} -1 \\ 0 \\ 1 \end{bmatrix} = 2\mathbf{b}_3$$

so that

$$\boxed{\Lambda = [T_A]_\beta = \begin{bmatrix} 4 & 0 & 0 \\ 0 & 2 & 0 \\ 0 & 0 & 2 \end{bmatrix}}$$

5. Expanding along the 4th row (you could also expand along the first column), we get that

$$f(x) = \det \begin{bmatrix} 1 & 1 & 2 & 3 & 4 \\ 9 & 0 & 2 & 3 & 4 \\ 9 & 0 & 0 & 3 & 4 \\ x & 1 & 2 & 9 & 1 \\ 7 & 0 & 0 & 0 & 4 \end{bmatrix} = -x\det(A_{4,1}) + \det(A_{4,2}) - 2\det(A_{4,3}) + 9\det(A_{4,4}) - \det(A_{4,5})$$

Note that the terms $\det(A_{4,j})$ are constants, so when we take the derivative of f we treat them as such, and therefore:

$$f'(x) = -\det(A_{4,1}) = -\det \begin{bmatrix} 1 & 2 & 3 & 4 \\ 0 & 2 & 3 & 4 \\ 0 & 0 & 3 & 4 \\ 0 & 0 & 0 & 4 \end{bmatrix} = -1 \cdot 2 \cdot 3 \cdot 4 = \boxed{-24}$$

6. Let $\beta = (1, x, x^2)$ and $\beta' = (1 + x + x^2,\ 1 + x,\ 1 - x^2)$. Then

$$M_{\beta',\beta} = \begin{bmatrix} [1 + x + x^2]_\beta & [1 + x]_\beta & [1 - x^2]_\beta \end{bmatrix} = \boxed{\begin{bmatrix} 1 & 1 & 1 \\ 1 & 1 & 0 \\ 1 & 0 & -1 \end{bmatrix}}$$

and therefore

$$M_{\beta,\beta'} = M_{\beta',\beta}^{-1} = \boxed{\begin{bmatrix} 1 & -1 & 1 \\ -1 & 2 & -1 \\ 1 & -1 & 0 \end{bmatrix}}$$

Finally,

$$[2x - 5x^2]_{\beta'} = M_{\beta,\beta'}[2x - 5x^2]_\beta = \begin{bmatrix} 1 & -1 & 1 \\ -1 & 2 & -1 \\ 1 & -1 & 0 \end{bmatrix} \begin{bmatrix} 2 \\ 0 \\ -5 \end{bmatrix} = \begin{bmatrix} -7 \\ 9 \\ -2 \end{bmatrix}$$

which means

$$\boxed{2x - 5x^2 = -7(1 + x + x^2) + 9(1 + x) - 2(1 - x^2)}$$

7. (a) $(AB)^T_{ij} = (AB)_{ji} = \sum_{k=1}^{n} A_{jk}B_{ki} = \sum_{k=1}^{n} B_{ki}A_{jk} = \sum_{k=1}^{n} (B^T)_{ik}(A^T)_{kj} = (B^T A^T)_{ij}$

   (b) If A is invertible, then there is a matrix A^{-1} such that $AA^{-1} = A^{-1}A = I$, whence by part (a) of this problem $(A^{-1})^T A^T = (AA^{-1})^T = I^T = I = I^T = (A^{-1}A)^T = A^T(A^{-1})^T$. This shows that A^T is invertible, and $(A^{-1})^T = (A^T)^{-1}$.

8. Let $A = \begin{bmatrix} 0 & -2 \\ 1 & 3 \end{bmatrix}$.

   (a) $c_A(x) = \det(A - xI_2) = \det \begin{bmatrix} -x & -2 \\ 1 & 3 - x \end{bmatrix} = x^2 - 3x + 2 = (x - 1)(x - 2)$. Thus $\lambda_1 = 1$ and $\lambda_2 = 2$ are the eigenvalues of A. Let's find the dimensions of their corresponding eigenspaces:

$$E_1 = \ker(A - I_2) = \ker \begin{bmatrix} -1 & -2 \\ 1 & 2 \end{bmatrix} = \ker \begin{bmatrix} 1 & 2 \\ 0 & 0 \end{bmatrix}$$

has dimension 1, and

$$E_2 = \ker(A - 2I_2) = \ker \begin{bmatrix} -2 & -2 \\ 1 & 1 \end{bmatrix} = \ker \begin{bmatrix} 1 & 1 \\ 0 & 0 \end{bmatrix}$$

also has dimension 1. Thus, since the dimensions of the eigenspaces equal the algebraic multiplicities of the corresponding eigenvalues, A is diagonalizable.

(b) First, to find bases for E_1 and E_2, we parametrize the free, or inner, variable y: for E_1 this means letting $y = t$ and so $x = -2t$, while for E_2 this means $y = t$ and $x = -t$, so that a basis for E_1 is $(-2, 1)$, and a basis for E_2 is $(-1, 1)$. Joining these two bases gives a basis for $\mathbb{R}^2$ that diagonalizes A:

$$\beta = \left\{ \begin{bmatrix} -2 \\ 1 \end{bmatrix}, \begin{bmatrix} -1 \\ 1 \end{bmatrix} \right\}$$

Then an easy calculation shows that $A\mathbf{b}_1 = \mathbf{b}_1$ and $A\mathbf{b}_2 = 2\mathbf{b}_2$, so

$$\boxed{\Lambda = [T_A]_\beta \begin{bmatrix} 1 & 0 \\ 0 & 2 \end{bmatrix}}$$

Finally, with ρ the standard basis for $\mathbb{R}^2$,

$$M = M_{\beta,\rho} = \left[\begin{bmatrix} -2 \\ 1 \end{bmatrix}_\rho \; \begin{bmatrix} -1 \\ 1 \end{bmatrix}_\rho \right] = \boxed{\begin{bmatrix} -2 & -1 \\ 1 & 1 \end{bmatrix}}$$

so that

$$M^{-1} = \begin{bmatrix} -1 & -1 \\ 1 & 2 \end{bmatrix}$$

and an easy verification shows that $A = M\Lambda M^{-1}$.

9. $\boxed{\det A = -3}$, which can be computed by using Gaussian elimination or by cofactor expansion.

10. If $A \in \mathbb{R}^{5\times 5}$ and $\det(A) = 5$, then $\boxed{\det(-2A) = (-2)^5 \det A = -32 \cdot 5 = -160}$.