Linear Algebra Midterm
Summer 2011

1. Consider the following system of four equations in five unknowns:

$$\begin{aligned}
3x_1 - x_2 + 3x_3 - x_4 + 2x_5 &= 5 \\
x_1 - x_2 - x_3 - 2x_4 - x_5 &= 2 \\
5x_1 - 2x_2 + x_3 - 3x_4 + 3x_5 &= 10 \\
2x_1 - x_2 \phantom{{}+x_3} - 2x_4 + x_5 &= 5
\end{aligned}$$

   (a) If A is the coefficient matrix of the system, find a basis β for $\ker A$. (You'll need to consider $\mathrm{rref}([A|\mathbf{0}])$ here.)

   (b) Find the set of all solutions $K = \mathbf{s} + \ker A = \mathbf{s} + \mathrm{span}(\beta)$.

(c) Use the previous result to find two distinct solutions, and verify that they solve the system.

2. Let $S = \{2 + x + x^2 + 3x^3,\ 4 + 2x + 4x^2 + 6x^3,\ 6 + 3x + 8x^2 + 7x^3,\ 2 + x + 5x^3,\ 4 + x + 9x^3\}$ and let $V = \text{span}(S)$ be the subspace of $\mathbb{R}_3[x]$ generated by S. Find the dimension of V by reducing (if necessary) S to a basis β for V.

3. Show that the matrices $A = \begin{bmatrix} 0 & 1 \\ 0 & 0 \end{bmatrix}$ and $B = \begin{bmatrix} 0 & 1 \\ -1 & 0 \end{bmatrix}$ are not diagonalizable.

4. Let

$$A = \begin{bmatrix} 3 & 1 & 1 \\ 2 & 4 & 2 \\ -1 & -1 & 1 \end{bmatrix}$$

Show that A is diagonalizable, then find a basis β for $\mathbb{R}^3$ such that the associated operator $T_A \in \mathcal{L}(\mathbb{R}^3)$ has a diagonal matrix representation $[T_A]_\beta$ with respect to β, and finally compute $[T_A]_\beta$.

5. Find the derivative of the function $f : \mathbb{R} \to \mathbb{R}$ given by

$$f(x) = \begin{bmatrix} 1 & 1 & 2 & 3 & 4 \\ 9 & 0 & 2 & 3 & 4 \\ 9 & 0 & 0 & 3 & 4 \\ x & 1 & 2 & 9 & 1 \\ 7 & 0 & 0 & 0 & 4 \end{bmatrix}$$

6. Let $\beta = (1, x, x^2)$ and $\beta' = (1 + x + x^2,\ 1 + x,\ 1 - x^2)$ be two ordered bases for $\mathbb{R}_2[x]$. Find $M_{\beta',\beta}$, the matrix changing β' coordinates into standard β coordinates, and find its inverse $M^{-1}_{\beta',\beta} = M_{\beta,\beta'}$. Then use the result to express the polynomial $2x - 5x^2$ in β' coordinates.

7. (a) Let $A, B \in \mathbb{R}^{n \times n}$. Show that $(AB)^T = B^T A^T$. [Hint: Show that the ijth elements of each are equal, $(AB)^T_{ij} = (B^T A^T)_{ij}$.]

   (b) Show that if $A \in \mathbb{R}^{n \times n}$ is invertible, then A^T is also invertible, and $(A^T)^{-1} = (A^{-1})^T$.

8. Let $A = \begin{bmatrix} 0 & -2 \\ 1 & 3 \end{bmatrix}$.

   (a) Show that A is diagonalizable.

(b) Find the diagonal matrix $\Lambda = [T_A]_\beta$, and find an invertible matrix M such that $A = M\Lambda M^{-1}$. Compute this product explicitly to verify the equation.

9. Compute $\det A$, where

$$A = \begin{bmatrix} 0 & 2 & 1 & 3 \\ 1 & 0 & -2 & 2 \\ 3 & -1 & 0 & 1 \\ -1 & 1 & 2 & 0 \end{bmatrix}$$

10. Suppose that $A \in \mathbb{R}^{5\times 5}$ and $\det(A) = 5$. What is $\det(-2A)$?