Examples and Exercises

0.1 Worked Examples

Example 0.1 *In* Example 0.20, Matrix Representations, *we showed that the linear transformation* $T \in \mathcal{L}(\mathbb{R}^2, \mathbb{R}^3)$ *given by* $T(x, y) = (x + y, 2x - y, 3x + 5y)$, *which has a matrix representation with respect to standard bases* $\rho_2 = (\mathbf{e}_1, \mathbf{e}_2)$ *and* $\rho_3 = (\mathbf{e}_1, \mathbf{e}_2, \mathbf{e}_3)$

$$[T]_{\rho_2}^{\rho_3} = \begin{bmatrix} 1 & 1 \\ 2 & -1 \\ 3 & 5 \end{bmatrix}$$

has a different matrix representation with respect to the bases $\beta = (\mathbf{v}_1, \mathbf{v}_2) = \big((1, 1), (0, -1)\big)$ *and* $\gamma = (\mathbf{w}_1, \mathbf{w}_2, \mathbf{w}_3) = \big((1, 1, 1), (1, 0, 1), (0, 0, 1)\big)$, *namely*

$$[T]_{\beta}^{\gamma} = \begin{bmatrix} 1 & 1 \\ 1 & -2 \\ 6 & -4 \end{bmatrix}$$

We achieved this second representation by computing $[T(\mathbf{v}_i)]_\gamma$, *namely,* $[T(\mathbf{v}_i)]_\gamma = C^{-1}T(\mathbf{v}_i)$, *where*

$$C = \begin{bmatrix} \mathbf{w}_1 & \mathbf{w}_2 & \mathbf{w}_3 \end{bmatrix} = \begin{bmatrix} 1 & 1 & 0 \\ 1 & 0 & 0 \\ 1 & 1 & 1 \end{bmatrix} \qquad \text{so that} \qquad C^{-1} = \begin{bmatrix} 0 & 1 & 0 \\ 1 & -1 & 0 \\ -1 & 0 & 1 \end{bmatrix}$$

Let

$$B = \begin{bmatrix} \mathbf{v}_1 & \mathbf{v}_2 \end{bmatrix} = \begin{bmatrix} 1 & 0 \\ 1 & -1 \end{bmatrix} \qquad \text{so that} \qquad B^{-1} = B = \begin{bmatrix} 1 & 0 \\ 1 & -1 \end{bmatrix}$$

Show that this representation may also be derived by using Theorem 0.23 *of* Matrix Representations, *namely, by using the relation:*

$$C^{-1}[T]_{\rho_2}^{\rho_3} = [T]_{\beta}^{\gamma} B^{-1}$$

Solution: This is easy enough, since $C^{-1}[T]_{\rho_2}^{\rho_3} = [T]_{\beta}^{\gamma} B^{-1}$ holds iff $C^{-1}[T]_{\rho_2}^{\rho_3} B = [T]_{\beta}^{\gamma}$, just right-multiply by B! Well, let's do this:

$$\begin{aligned} [T]_{\beta}^{\gamma} &= C^{-1}[T]_{\rho_2}^{\rho_3} B \\ &= \begin{bmatrix} 0 & 1 & 0 \\ 1 & -1 & 0 \\ -1 & 0 & 1 \end{bmatrix} \begin{bmatrix} 1 & 1 \\ 2 & -1 \\ 3 & 5 \end{bmatrix} \begin{bmatrix} 1 & 0 \\ 1 & -1 \end{bmatrix} \\ &= \begin{bmatrix} 0 & 1 & 0 \\ 1 & -1 & 0 \\ -1 & 0 & 1 \end{bmatrix} \begin{bmatrix} 2 & -1 \\ 1 & 1 \\ 8 & -5 \end{bmatrix} \\ &= \begin{bmatrix} 1 & 1 \\ 1 & -2 \\ 6 & -4 \end{bmatrix} \end{aligned}$$

This verifies the claim of that theorem for this example, and it exemplifies the second method of attack for such a problem. ■

Example 0.2 *If $T \in \mathcal{L}(\mathbb{R}_3[x], \mathbb{R}_2[x])$ is given by $T(f(x)) = f'(x)$, and $\beta = (1, x, x^2, x^3)$ and $\gamma = (1, x, x^2)$ are the standard ordered bases for $\mathbb{R}_3[x]$ and $\mathbb{R}_2[x]$, respectively, the matrix representation of T is found as follows:*

$$\left.\begin{aligned} T(1) &= 1' = 0 && = 0 \cdot 1 + 0x + 0x^2 \\ T(x) &= x' = 1 && = 1 \cdot 1 + 0x + 0x^2 \\ T(x^2) &= (x^2)' = 2x && = 0 \cdot 1 + 2x + 0x^2 \\ T(x^3) &= (x^3)' = 3x^2 && = 0 \cdot 1 + 0x + 3x^2 \end{aligned}\right\} \implies [T]_\beta^\gamma = \begin{bmatrix} 0 & 1 & 0 & 0 \\ 0 & 0 & 2 & 0 \\ 0 & 0 & 0 & 3 \end{bmatrix} \qquad \blacksquare$$

Example 0.3 *Let $T \in \mathcal{L}\big(\mathbb{R}_3[x], \mathbb{R}_2[x]\big)$ be given by $T(f(x)) = f'(x)$ and let β and γ be the standard ordered bases for $\mathbb{R}_3[x]$ and $\mathbb{R}_2[x]$, respectively. Verify that*

$$[T(p(x))]_\gamma = [T]_\beta^\gamma [p(x)]_\beta$$

for the polynomial $p(x) = 2 - 4x + x^2 + 3x^3 \in \mathbb{R}_3[x]$.

Solution: First, for $p(x) = 2 - 4x + x^2 + 3x^3$ we have $T(p(x)) = p'(x) = -4 + 2x + 9x^2$. Moreover, by Example 0.2 we have

$$[T]_\beta^\gamma = \begin{bmatrix} 0 & 1 & 0 & 0 \\ 0 & 0 & 2 & 0 \\ 0 & 0 & 0 & 3 \end{bmatrix}$$

and clearly we have

$$[p(x)]_\beta = \begin{bmatrix} 2 \\ -4 \\ 1 \\ 3 \end{bmatrix} \qquad [T(p(x))]_\gamma = [p'(x)]_\gamma = \begin{bmatrix} -4 \\ 2 \\ 9 \end{bmatrix}$$

Consequently,

$$[T]_\beta^\gamma [p(x)]_\beta = \begin{bmatrix} 0 & 1 & 0 & 0 \\ 0 & 0 & 2 & 0 \\ 0 & 0 & 0 & 3 \end{bmatrix} \begin{bmatrix} 2 \\ -4 \\ 1 \\ 3 \end{bmatrix} = \begin{bmatrix} -4 \\ 2 \\ 9 \end{bmatrix} = [T(p(x))]_\gamma \qquad \blacksquare$$

Example 0.4 *Let $T \in \mathcal{L}\big(\mathbb{R}^{2\times 2}, \mathbb{R}^{2\times 2}\big)$ be given by $T(A) = A^T$, and let*

$$B = \{e_1, e_2, e_3, e_4\} = \left\{ \begin{pmatrix} 1 & 0 \\ 0 & 0 \end{pmatrix}, \begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix}, \begin{pmatrix} 0 & 0 \\ 1 & 0 \end{pmatrix}, \begin{pmatrix} 0 & 0 \\ 0 & 1 \end{pmatrix} \right\}$$

be a basis for $\mathbb{R}^{2\times 2}$. Verify that $[T(A)]_B = [T]_B [A]_B$, for

$$A = \begin{pmatrix} 1 & 4 \\ -1 & 6 \end{pmatrix}$$

Solution: First,

$$[T(A)]_B = [A^T]_B = \left[\begin{pmatrix} 1 & -1 \\ 4 & 6 \end{pmatrix}\right]_B = [1e_1 - 1e_2 + 4e_3 + 6e_4]_B = \begin{pmatrix} 1 \\ -1 \\ 4 \\ 6 \end{pmatrix}$$

Obviously T is linear, because it is componentwise linear in $\mathbb{R}$. We next need to determine $[T]_B$ and $[A]_B$:

$$\left.\begin{aligned} T(e_1) &= e_1 = 1e_1 + 0e_2 + 0e_3 + 0e_4 \\ T(e_2) &= e_3 = 0e_1 + 0e_2 + 1e_3 + 0e_4 \\ T(e_3) &= e_2 = 0e_1 + 1e_2 + 0e_3 + 0e_4 \\ T(e_4) &= e_4 = 0e_1 + 0e_2 + 0e_3 + 1e_4 \end{aligned}\right\} \quad \Longrightarrow \quad \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix}$$

Consequently

$$[T]_B[A]_B = \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix} \begin{pmatrix} 1 \\ 4 \\ -1 \\ 6 \end{pmatrix} = \begin{pmatrix} 1 \\ -1 \\ 4 \\ 6 \end{pmatrix} = [T(A)]_B \qquad \blacksquare$$

Example 0.5 *Given distinct constants $c_0, c_1, \ldots, c_n \in \mathbb{R}$, we can define $n+1$ distinct polynomials that each have all except one of the $c_0, \ldots, c_n$ as roots, and such that they are linearly independent as vectors in $\mathbb{R}_{n+1}[x]$. This gives us a special basis for $\mathbb{R}_{n+1}[x]$, one which allows the construction of an $n+1$-degree polynomial passing through the points $(c_0, b_0), \ldots, (c_n, b_n) \in \mathbb{R}^2$. The* **Lagrange polynomials**, *$\ell_0(x), \ldots, \ell_n(x) \in \mathbb{R}_{n+1}[x]$ associated with the $c_0, \ldots, c_n$ are given by*

$$\ell_i(x) = \frac{(x-c_0)\cdots(x-c_{i-1})(x-c_{i+1})\cdots(x-c_n)}{(c_i-c_0)\cdots(c_i-c_{i-1})(c_i-c_{i+1})\cdots(c_i-c_n)} = \prod_{\substack{k=0\\k\neq i}}^{n} \frac{x-c_k}{c_i-c_k}$$

As a consequence of this definition we have that

$$\ell_i(c_j) = \begin{cases} 0, & \text{if } i \neq j \\ 1, & \text{if } i = j \end{cases} \tag{0.1}$$

This property of Lagrange polynomials means that $\beta = \{\ell_0, \ell_1, \ldots, \ell_n\}$ is a basis for $\mathbb{R}_{n+1}[x]$. As a result, if we specify a set of $n+1$ constants in $b_0, b_1, \ldots, b_n \in \mathbb{R}$, the polynomial

$$g(x) = \sum_{i=0}^{n} b_i \ell_i(x)$$

is the unique polynomial in $\mathbb{R}[x]$ such that $g(c_j) = b_j$ for all $j = 0, 1, \ldots, n$. The process of finding the polynomial $g(x)$ such that $g(c_j) = b_j$ for all $j = 0, 1, \ldots, n$ is called **Lagrange interpolation**.

This is shown as follows: First, note that $\beta = \{\ell_0, \ell_1, \ldots, \ell_n\}$ is a linearly independent subset of $\mathbb{R}_n[x]$, since if we choose scalars $a_0, \ldots, a_n \in \mathbb{R}$, and we let $p_0(x) \equiv 0$ (the zero polynomial), and if we suppose

$$\begin{aligned} a_0\ell_0(x) + a_1\ell_1(x) + \cdots + a_n\ell_n(x) &= p_0(x) \\ &= 0 \end{aligned} \qquad \text{for all } x \in \mathbb{R}$$

we have by (0.1)

$$\begin{aligned} a_0\ell(c_j) + a_1\ell_1(c_j) + \cdots + a_n\ell_n(c_j) &= a_j\ell_j(c_j) \\ &= a_j \\ &= 0 \end{aligned} \qquad \text{for } \; j = 0, 1, \ldots, n$$

so $a_0 = \cdots = a_n = 0$ and β is linearly independent. Since $\dim\big(\mathbb{R}_n[x]\big) = n+1$, it follows that β is a basis for $\mathbb{R}_n[x]$. Consequently, every n-th degree polynomial in $\mathbb{R}_n[x]$ is a linear combination of polynomials in β, so that if $g \in \mathbb{R}_n[x]$, then $\exists b_1, \ldots, b_n \in \mathbb{R}$ such that

$$g = b_1\ell_1 + \cdots + b_n\ell_n$$

By (0.1) *this means that for* $j = 1, \ldots, n$

$$g(c_j) = b_1\ell_1(c_j) + \cdots + b_n\ell_n(c_j) = b_j\ell(c_j) = b_j$$

so that

$$g(x) = g(c_0)\ell_0(x) + g(c_1)\ell_1(x) + \cdots + g(c_n)\ell_n(x) \tag{0.2}$$

is the unique representation of g *as a linear combination of elements of* β. (0.2) *is called the* **Lagrange interpolation formula**. ■

Example 0.6 *Find the real polynomial* $g(x) \in \mathbb{R}[x]$ *of degree at most* 2 *whose graph contains the points* $(1, 8)$, $(2, 5)$, *and* $(3, -4)$.

Solution: $(1, 8), (2, 5), (3, -4) \in G(g) \implies g(1) = 8$, $g(2) = 5$ and $g(3) = -4$, so that $c_0 = 1$, $c_1 = 2$, $c_2 = 3$, $b_0 = 8$, $b_1 = 5$ and $b_2 = -4$. Hence

$$\begin{aligned}
\ell_0(x) &= \frac{(x - c_1)(x - c_2)}{(c_0 - c_1)(c_0 - c_2)} = \frac{(x - 2)(x - 3)}{(1 - 2)(1 - 3)} = \frac{1}{2}(x^2 - 5x + 6) \\
\ell_1(x) &= \frac{(x - c_0)(x - c_2)}{(c_1 - c_0)(c_1 - c_2)} = \frac{(x - 1)(x - 3)}{(2 - 1)(2 - 3)} = -x^2 + 4x - 3 \\
\ell_2(x) &= \frac{(x - c_0)(x - c_1)}{(c_2 - c_0)(c_2 - c_1)} = \frac{(x - 1)(x - 2)}{(3 - 1)(3 - 2)} = \frac{1}{2}(x^2 - 3x + 2)
\end{aligned}$$

so that by the previous example,

$$\begin{aligned}
g(x) &= b_0\ell_0(x) + b_1\ell_1(x) + b_2\ell_2(x) \\
&= 8\Big[\frac{1}{2}(x^2 - 5x + 6)\Big] + 5\Big[-x^2 + 4x - 3\Big] - 4\Big[\frac{1}{2}(x^2 - 3x + 2)\Big] \\
&= (4 - 5 - 2)x^2 + (-20 + 20 + 6)x + 24 - 15 - 4 \\
&= -3x^2 + 6x + 5
\end{aligned}$$

which is the desired polynomial. ■

Example 0.7 *Verify directly that* $\mathbb{R}_3[x] \cong \mathbb{R}^{2\times 2}$.

Solution: We give here a rather novel way of doing it. Define $T \in \mathcal{L}\big(\mathbb{R}[x], M_2(\mathbb{R})\big)$ by

$$T(f(x)) = \begin{pmatrix} f(1) & f(2) \\ f(3) & f(4) \end{pmatrix}$$

Then, T is indeed linear:

$$\begin{aligned}
T(af(x) + bg(x)) &= \begin{pmatrix} (af + bg)(1) & (af + bg)(2) \\ (af + bg)(3) & (af + bg)(4) \end{pmatrix} \\
&= \begin{pmatrix} af(1) + bg(1) & af(2) + bg(2) \\ af(3) + bg(3) & af(4) + bg(4) \end{pmatrix} \\
&= a\begin{pmatrix} f(1) & f(2) \\ f(3) & f(4) \end{pmatrix} + b\begin{pmatrix} g(1) & g(2) \\ g(3) & g(4) \end{pmatrix} \\
&= aT(f(x)) + bT(g(x))
\end{aligned}$$

Moreover, T is 1-1: if $T(f(x)) = T(g(x))$, then

$$\begin{pmatrix} f(1) & f(2) \\ f(3) & f(4) \end{pmatrix} = \begin{pmatrix} g(1) & g(2) \\ g(3) & g(4) \end{pmatrix}$$

whence $f(1) = g(1)$, $f(2) = g(2)$, $f(3) = g(3)$ and $f(4) = g(4)$, whence by Example 0.5 (Lagrange interpolation),

$$\begin{aligned} f(x) &= f(1)p_1(x) + f(2)p_2(x) + f(3)p_3(x) + f(4)p_4(x) \\ &= g(1)p_1(x) + g(2)p_2(x) + g(3)p_3(x) + g(4)p_4(x) \\ &= g(x) \end{aligned}$$

where $p_1(x), \ldots, p_4(x)$ are Lagrange polynomials in $\mathbb{R}_3[x]$ associated with 1, 2, 3 and 4. Alternatively,

$$T(f(x)) = O \iff f(1) = f(2) = f(3) = f(4) = 0$$

in which case $f(x) = f_0(x)$, the zero polynomial, which also follows from the Lagrange polynomial method of Example 0.5, so that $\ker(T) = \{f_0(x)\}$. Hence, T is 1-1. But T is also onto: note that $\dim(\mathbb{R}[x]) = 4 = \dim(\mathbb{R}^{2\times 2})$, so we can deduce that T is onto, whence it is bijective and invertible, and so an isomorphism. Therefore $\mathbb{R}_3[x] \cong \mathbb{R}^{2\times 2}$. ∎

Example 0.8 *Show that equation $2x^2 - 4xy + 5y^2 = 1$ is the equation of an ellipse rotated through an angle of $\pi/6$ radians.*

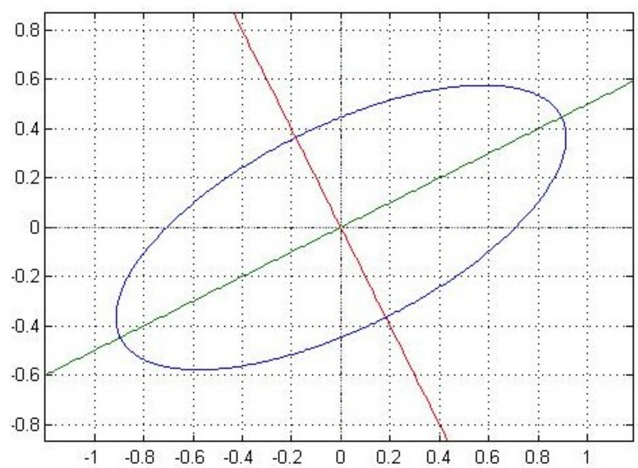

Figure 0.1: $2x^2 - 4xy + 5y^2 = 1$, the ellipse $x^2 + 6y^2 = 1$ rotated through $\pi/6$.

Suppose we want to rotate the ellipse $x^2 + 6y^2 = 1$ by $\pi/6$. We regard this equation as the pre-image or level set of 0

$$F^{-1}(0)$$

under the function $F : \mathbb{R}^2 \to \mathbb{R}$ given by

$$F(\mathbf{x}) = F(x, y) = x^2 + 6y^2 - 1$$

What we want to do is change coordinates in $\mathbb{R}^2$, that is change bases in $\mathbb{R}^2$, in such a way that the old basis is orthonormal but rotated through an angle of $\pi/6$, and the function $F(\mathbf{x}') = F(x', y') = 0$ is the same expression but in the old coordinates x', y'. When we translate this expression from x', y' coordinates into x, y, the standard Castesian coordinates, we'll get a new expression in the regular Cartesian coordinates, representing the rotated ellipse. Thus the problem reduces to finding $\mathbf{x}' = [\mathbf{x}']_\beta$ in terms of $\mathbf{x} = [\mathbf{x}]_\gamma$, where γ is the old basis and $\beta = \{\mathbf{e}_1, \mathbf{e}_2\}$ is the new standard basis. This reduces to finding the change of basis matrix, $M_{\gamma,\beta}$,

$$\mathbf{x}' = [\mathbf{x}']_\beta = M_{\gamma,\beta}[\mathbf{x}]_\gamma$$

and inserting the result into $F(\mathbf{x}') = 0$. Towards this end, we first choose orthogonal vectors $\mathbf{x}'_1$ and $\mathbf{x}'_2$ such that $\mathbf{x}'_1$ is rotated $\pi/6$ from the x-axis and $\mathbf{x}'_2$ is rotated $\pi/6$ from the y-axis. An obvious choice is $\mathbf{x}'_1 = (2,1)$ and $\mathbf{x}'_2 = (-1,2)$, since $\sin(\pi/6) = 1/2$. If we multiply each by $1/|\mathbf{x}'_1| = 1/|\mathbf{x}'_2| = 1/\sqrt{5}$, then we have orthonormal vectors $\mathbf{v}'_1$ and $\mathbf{v}'_2$ which give the old basis

$$\gamma = \{\mathbf{v}'_1, \mathbf{v}'_2\} = \Big\{(2/\sqrt{5},\ 1/\sqrt{5}), (-1/\sqrt{5},\ 2/\sqrt{5})\Big\}$$

We now calculate the change of coordinates matrix $M_{\gamma,\beta}$:

$$\phi_\beta(\mathbf{v}_1) = \begin{bmatrix} \frac{2}{\sqrt{5}} \\ \frac{1}{\sqrt{5}} \end{bmatrix}_\beta = \begin{pmatrix} \frac{2}{\sqrt{5}} \\ \frac{1}{\sqrt{5}} \end{pmatrix} \quad \text{and} \quad \phi_\beta(\mathbf{v}_2) = \begin{bmatrix} \frac{-1}{\sqrt{5}} \\ \frac{2}{\sqrt{5}} \end{bmatrix}_\beta = \begin{pmatrix} \frac{-1}{\sqrt{5}} \\ \frac{2}{\sqrt{5}} \end{pmatrix}$$

so that

$$M_{\beta,\gamma}^{-1} = M_{\gamma,\beta} = \begin{bmatrix} \phi_\beta(\mathbf{v}_1) & \phi_\beta(\mathbf{v}_2) \end{bmatrix} = \begin{pmatrix} \frac{2}{\sqrt{5}} & \frac{-1}{\sqrt{5}} \\ \frac{1}{\sqrt{5}} & \frac{2}{\sqrt{5}} \end{pmatrix}$$

Therefore,

$$\mathbf{x}' = \begin{pmatrix} x' \\ y' \end{pmatrix} = [\mathbf{x}']_\beta = M_{\gamma,\beta}[\mathbf{x}]_\gamma = \begin{pmatrix} \frac{2}{\sqrt{5}} & \frac{-1}{\sqrt{5}} \\ \frac{1}{\sqrt{5}} & \frac{2}{\sqrt{5}} \end{pmatrix} \begin{pmatrix} x \\ y \end{pmatrix} = \begin{pmatrix} \frac{2}{\sqrt{5}}x - \frac{1}{\sqrt{5}}y \\ \frac{1}{\sqrt{5}}x + \frac{2}{\sqrt{5}}y \end{pmatrix}$$

or

$$x' = \frac{2}{\sqrt{5}}x - \frac{1}{\sqrt{5}}y \qquad\qquad y' = \frac{1}{\sqrt{5}}x + \frac{2}{\sqrt{5}}y$$

Consequently,

$$\begin{aligned} (F \circ \phi_\beta)(\mathbf{x}') &= \left(\frac{2}{\sqrt{5}}x - \frac{1}{\sqrt{5}}y\right)^2 + 6\left(\frac{1}{\sqrt{5}}x + \frac{2}{\sqrt{5}}y\right)^2 - 1 \\ &= \left(\frac{4}{5} + \frac{6}{5}\right)x^2 + \left(-\frac{4}{5} + \frac{24}{5}\right)xy + \left(\frac{1}{5} + \frac{24}{5}\right)y^2 - 1 \\ &= 2x^2 + 4xy + 5y^2 - 1 \\ &= 0 \end{aligned}$$

and we have our equation, $2x^2 - 4xy + 5y^2 = 1$, in standard coordinates. I graphed the ellipse with Matlab. First, I started with the simple equation $x^2 + 6y^2 = 1$ in the old basis, then I parametrized x and y by introducing the function $f : [0, 2\pi] \to \mathbb{R}^2$ given by

$$f(t) = \mathbf{x}(t) = \begin{pmatrix} x(t) \\ y(t) \end{pmatrix} = \begin{pmatrix} \cos(t) \\ \frac{1}{\sqrt{6}}\sin(t) \end{pmatrix}$$

since from the equation $x^2 + 6y^2 = 1$ we are given that $\frac{1}{a^2} = 1$ and $\frac{1}{b^2} = 6$, or $a = 1$ and $b = \frac{1}{\sqrt{6}}$. Then I used $M_{\gamma,\beta}$ to get the new coordinates, which I labeled z and w, and which are given by

$$z = \frac{2}{\sqrt{5}}x - \frac{1}{\sqrt{5}}y \qquad \text{and} \qquad w = \frac{1}{\sqrt{5}}x + \frac{2}{\sqrt{5}}y$$

That is, I composed g with f, where $g : \mathbb{R}^2 \to \mathbb{R}^2$ is given by $g(\mathbf{x}) = M_{\gamma,\beta}\mathbf{x}$, so that what we end up having is $g \circ f : [0, 2\pi] \to \mathbb{R}^2$ given by

$$(g \circ f)(t) \;=\; M_{\gamma,\beta}\mathbf{x}(t) \;=\; \begin{pmatrix} \frac{2}{\sqrt{5}} & \frac{-1}{\sqrt{5}} \\ \frac{1}{\sqrt{5}} & \frac{2}{\sqrt{5}} \end{pmatrix} \begin{pmatrix} \cos(t) \\ \frac{1}{\sqrt{6}}\sin(t) \end{pmatrix} \;=\; \begin{pmatrix} \frac{2}{\sqrt{5}}\cos(t) - \frac{1}{\sqrt{30}}\sin(t) \\ \frac{1}{\sqrt{5}}\cos(t) + \frac{2}{\sqrt{30}}\sin(t) \end{pmatrix}$$

The figure can then be generated with either of the following Matlab codes:

```
t=[0:0.01:2*pi];
x=cos(t);
y=sin(t)./sqrt(6);
z=2.*x./sqrt(5)-y./sqrt(5);
w=x./sqrt(5)+2.*y./sqrt(5);
a=[-1.5:0.01:1.5]; b=[-1:0.01:1];
c=0; d=0; e=a./2; f=-2*a;
plot(z,w,a,e,a,f,a,c,d,b)
axis equal
grid on
```

or

```
t=[0:0.01:2*pi];
x=2.*cos(t)./sqrt(5)-sin(t)./sqrt(30);
y=cos(t)./sqrt(5)+2.*sin(t)./sqrt(30);
a=[-1.5:0.01:1.5]; b=[-1:0.01:1];
c=0; d=0; e=a./2; f=-2*a;
plot(x,y,a,e,a,f,a,c,d,b)
axis equal
grid on
```

■

Example 0.9 $4x^2 + 4xy + y^2 + \sqrt{5}x - 2\sqrt{5}y = 0$ *is the parabola* $y = x^2$ *rotated through an angle of* $\pi/6$ *radians.*

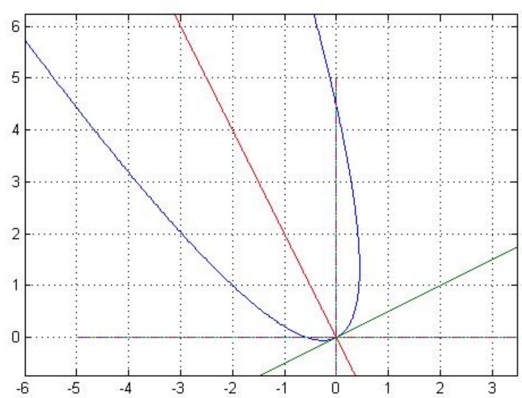

Figure 0.2: $4x^2 + 4xy + y^2 + \sqrt{5}x - 2\sqrt{5}y = 0$, the parabola $y = x^2$ rotated through $\pi/6$.

By the same procedure as in the previous example, starting with $y' = x'^2$ in the old coordinates, we get

$$x' = \frac{2}{\sqrt{5}}x + \frac{1}{\sqrt{5}}y \qquad \text{and} \qquad y' = \frac{-1}{\sqrt{5}}x + \frac{2}{\sqrt{5}}y$$

Plugging these into the original equation gives

$$\begin{aligned} y' = x'^2 \quad &\Longrightarrow \quad \frac{-1}{\sqrt{5}}x + \frac{2}{\sqrt{5}}y = \left(\frac{2}{\sqrt{5}}x + \frac{1}{\sqrt{5}}y\right)^2 \\ &\Longrightarrow \quad \frac{-1}{\sqrt{5}}x + \frac{2}{\sqrt{5}}y = \frac{4}{5}x^2 + \frac{4}{5}xy + \frac{1}{5}y^2 \\ &\Longrightarrow \quad 4x^2 + 4xy + y^2 + \sqrt{5}x - 2\sqrt{5}y = 0 \end{aligned}$$

To generate this figure I used the same matrix $M_{\gamma,\beta}$ as in the previous example, and the following Matlab code:

```
x=[-3:0.01:3];
y=x.^2;
z=2.*x./sqrt(5)-y./sqrt(5);
w=x./sqrt(5)+2.*y./sqrt(5);
a=[-5:0.01:5]; b=[-5:0.01:5];
c=0; d=0; e=a./2; f=-2*a;
plot(z,w,a,e,a,f,a,c,d,b)
axis equal
grid on
```

■

Example 0.10 *Determine whether the linear transformation $T \in \mathcal{L}\big(\mathbb{R}_2[x]\big)$ given by*

$$T(f(x)) = f(x) + f'(x) + f''(x)$$

is an isomorphism, and if so, what the inverse is.

Solution: First, we need to compute $[T]_\beta$, where β is the standard ordered basis of $\mathbb{R}_2[x]$.

$$\begin{aligned} T(1) &= 1 + 1' + 1'' = 1 + 0 + 0 = 1 \\ T(x) &= x + x' + x'' = x + 1 + 0 = 1 + x \\ T(x^2) &= x^2 + (x^2)' + (x^2)'' = x^2 + 2x + 2 \end{aligned} \implies [T]_\beta = \begin{pmatrix} 1 & 1 & 2 \\ 0 & 1 & 2 \\ 0 & 0 & 1 \end{pmatrix}$$

Now, $[T]_\beta$ is invertible since it's columns are linearly independent, so T is an isomorphism. We'll find $([T]_\beta)^{-1}$ by multiplication by elementary matrices, just to show how it's done (the full justification for this method is given below):

$$\begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & -2 \\ 0 & 0 & 1 \end{pmatrix} \left(\begin{array}{ccc|ccc} 1 & 1 & 2 & 1 & 0 & 0 \\ 0 & 1 & 2 & 0 & 1 & 0 \\ 0 & 0 & 1 & 0 & 0 & 1 \end{array}\right) = \left(\begin{array}{ccc|ccc} 1 & 1 & 2 & 1 & 0 & 0 \\ 0 & 1 & 0 & 0 & 1 & -2 \\ 0 & 0 & 1 & 0 & 0 & 1 \end{array}\right)$$

$$\implies \begin{pmatrix} 1 & 0 & -2 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix} \left(\begin{array}{ccc|ccc} 1 & 1 & 2 & 1 & 0 & 0 \\ 0 & 1 & 0 & 0 & 1 & -2 \\ 0 & 0 & 1 & 0 & 0 & 1 \end{array}\right) = \left(\begin{array}{ccc|ccc} 1 & 1 & 0 & 1 & 0 & -2 \\ 0 & 1 & 0 & 0 & 1 & -2 \\ 0 & 0 & 1 & 0 & 0 & 1 \end{array}\right)$$

$$\implies \begin{pmatrix} 1 & -1 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix} \left(\begin{array}{ccc|ccc} 1 & 1 & 0 & 1 & 0 & -2 \\ 0 & 1 & 0 & 0 & 1 & -2 \\ 0 & 0 & 1 & 0 & 0 & 1 \end{array}\right) = \left(\begin{array}{ccc|ccc} 1 & 0 & 0 & 1 & -1 & 0 \\ 0 & 1 & 0 & 0 & 1 & -2 \\ 0 & 0 & 1 & 0 & 0 & 1 \end{array}\right)$$

Now,

$$\begin{pmatrix} 1 & -1 & 0 \\ 0 & 1 & -2 \\ 0 & 0 & 1 \end{pmatrix} = \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & -2 \\ 0 & 0 & 1 \end{pmatrix} \begin{pmatrix} 1 & 0 & -2 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix} \begin{pmatrix} 1 & -1 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix}$$

so that the above procedure can be condensed into the following statement

$$\begin{pmatrix} 1 & -1 & 0 \\ 0 & 1 & -2 \\ 0 & 0 & 1 \end{pmatrix} \left(\begin{array}{ccc|ccc} 1 & 1 & 2 & 1 & 0 & 0 \\ 0 & 1 & 2 & 0 & 1 & 0 \\ 0 & 0 & 1 & 0 & 0 & 1 \end{array}\right) = \left(\begin{array}{ccc|ccc} 1 & 0 & 0 & 1 & -1 & 0 \\ 0 & 1 & 0 & 0 & 1 & -2 \\ 0 & 0 & 1 & 0 & 0 & 1 \end{array}\right)$$

which means

$$[T^{-1}]_\beta = ([T]_\beta)^{-1} = \begin{pmatrix} 1 & -1 & 0 \\ 0 & 1 & -2 \\ 0 & 0 & 1 \end{pmatrix}$$

so that

$$[T^{-1}(a+bx+cx^2)]_\beta = \begin{pmatrix}1 & -1 & 0\\ 0 & 1 & -2\\ 0 & 0 & 1\end{pmatrix}\begin{pmatrix}a\\ b\\ c\end{pmatrix} = \begin{pmatrix}a-b\\ b-2c\\ c\end{pmatrix}$$

Hence

$$\boxed{T^{-1}(a+bx+cx^2) = (a-b)+(b-2c)x+cx^2} \qquad \blacksquare$$

0.2 Exercises

1. Using the same T as in Example 0.1, but different β and γ, namely,

$$\beta = (\mathbf{v}_1, \mathbf{v}_2) = \big((1,2),(3,-5)\big), \qquad \gamma = (\mathbf{w}_1, \mathbf{w}_2, \mathbf{w}_3) = \big((1,-1,1),(2,3,1),(0,-2,1)\big)$$

compute $[T]_\beta^\gamma$. Do this in two different ways,

$$[T]_\beta^\gamma = \begin{bmatrix} C^{-1}T(\mathbf{v}_1) & C^{-1}T(\mathbf{v}_2) \end{bmatrix}$$

and

$$[T]_\beta^\gamma = C^{-1}[T]_{\rho_2}^{\rho_3}B$$

2. Let β and γ be as in Example 0.1 and let β' and γ' be as in Exercise 1 above. Compute $M_{\beta',\beta}$ and $M_{\gamma,\gamma'}$, and find $[T]_{\beta'}^{\gamma'}$ from $[T]_\beta^\gamma$, which we computed in Example 0.1, by using Theorem 0.30 in Matrix Representations, namely

$$[T]_{\beta'}^{\gamma'} = M_{\gamma,\gamma'}[T]_\beta^\gamma M_{\beta',\beta}$$

3. Let $\beta = (1, x, x^2)$ and $\beta' = (a_0 + a_1x + a_2x^2,\ b_0 + b_1x + b_2x^2,\ c_0 + c_1x + c_2x^2)$ be two ordered bases for $\mathbb{R}_2[x]$. Find $M_{\beta',\beta}$, the matrix changing β' coordinates into standard β coordinates, and find its inverse $M_{\beta',\beta}^{-1} = M_{\beta,\beta'}$. Then use the result to express the polynomial $2x - 5x^2$ in β' coordinates.

4. Show that $\mathbb{R}_3[x] \cong \mathbb{R}^{2\times 2}$ by using the function $T : \mathbb{R}_3[x] \to \mathbb{R}^{2\times 2}$ defined by

$$T\big(f(x)\big) = T\big(a + bx + cx^2 + dx^3\big) := \begin{pmatrix} a & b \\ c & d \end{pmatrix}$$

Show that T is linear, injective, and surjective, so that $T \in \mathrm{GL}(\mathbb{R}_3[x], \mathbb{R}^{2\times 2})$, i.e. T is an isomorphism.

5. Let $T : \mathbb{R}_1[x] \to \mathbb{R}_1[x]$ be defined by $T(a + bx) := (2a + b) + (a - 3b)x$. If we use the basis $\beta = \big(1 + x, 1 + 2x\big)$ for $\mathbb{R}_1[x]$, find T^{-1} by finding $([T]_\beta)^{-1}$.