Quiz 6

1. An $m \times n$ matrix

$$A = \begin{pmatrix} | & | & & | \\ \mathbf{a}_1 & \mathbf{a}_2 & \cdots & \mathbf{a}_n \\ | & | & & | \end{pmatrix}$$

can be viewed as a function from $\mathbb{R}^n$ to $\mathbb{R}^m$, sending $\mathbf{x} = \langle x_1, \ldots, x_n \rangle \in \mathbb{R}^n$ to $A\mathbf{x} = \sum_{i=1}^{n} x_i \mathbf{a}_i = \mathbf{b} \in \mathbb{R}^m$.

(a) True or False: If $m < n$, then as a function A cannot be one-to-one.

True: rref(A) here would have max m pivot cols., & therefore $n-m>0$ free variables, which would produce nontrivial solns.

(b) True or False: If $m > n$, then as a function A cannot be onto.

True: rref(A) would have at least one zero row, & more importantly would be missing the col. $\vec{e}_m = \begin{pmatrix} 0 \\ \vdots \\ 0 \\ 1 \end{pmatrix}$ which would never be achieved by $A\vec{x} = \vec{e}_m$ inconsistent)

(c) True or False: A is one-to-one iff ('if and only if') the columns of A are linearly independent.

True: A 1-1 $\overset{\text{def.}}{\iff}$ "$A\vec{x}_1 = A\vec{x}_2 \Rightarrow \vec{x}_1 = \vec{x}_2$" $\iff$ "$A\vec{x} = \vec{0} = A\vec{0}$ $\Rightarrow \vec{x} = \vec{0}$" in particular for $\vec{x}_1 = \vec{x}$, $\vec{x}_2 = \vec{0}$

(d) True or False: A is onto iff the columns of A span $\mathbb{R}^n$. ← typo

True: "For any $\vec{b} \in \mathbb{R}^m$, $A\vec{x} = \vec{b}$ is consistent." is what onto means. But $\vec{b} = A\vec{x} = x_1\vec{a}_1 + \cdots + x_n\vec{a}_n \in \text{span}(\vec{a}_1, \ldots, \vec{a}_n)$

(e) True or False: If $m = n$, then A, necessarily a square matrix, is invertible (both one-to-one and onto) iff $\mathbb{R}^n = \text{span}(\mathbf{a}_1, \ldots, \mathbf{a}_n)$.

True: $m = n \Rightarrow \text{rref}(A) = I = \begin{pmatrix} 1 & & 0 \\ & \ddots & \\ 0 & & 1 \end{pmatrix}$ in either case.