Quiz 5

1. Bring the following homogeneous system to row-reduced echelon form

$$\begin{aligned} 3x_1 + 3x_2 + 2x_3 + 4x_4 &= 0 \\ 6x_1 + 6x_2 + 2x_3 + 10x_4 &= 0 \end{aligned}$$

Then write the solution set as the span of vectors (the parameters coming from the free variables).

I'll work with the augmented coefficient matrix:

$$(A|\vec{0}) = \left(\begin{array}{cccc|c} 3 & 3 & 2 & 4 & 0 \\ 6 & 6 & 2 & 10 & 0 \end{array}\right)$$

$$\xrightarrow{-2I+II} \left(\begin{array}{cccc|c} 3 & 3 & 2 & 4 & 0 \\ 0 & 0 & -2 & 2 & 0 \end{array}\right)$$

$$\xrightarrow[-\frac{1}{2}II]{} \left(\begin{array}{cccc|c} 3 & 3 & 2 & 4 & 0 \\ 0 & 0 & 1 & -1 & 0 \end{array}\right)$$

$$\xrightarrow{-2II+I} \left(\begin{array}{cccc|c} 3 & 3 & 0 & 6 & 0 \\ 0 & 0 & 1 & -1 & 0 \end{array}\right)$$

$$\xrightarrow{\frac{1}{3}I} \left(\begin{array}{cccc|c} 1 & 1 & 0 & 2 & 0 \\ 0 & 0 & 1 & -1 & 0 \end{array}\right) = \mathrm{rref}(A|\vec{0})$$

i.e.

$$\begin{aligned} x_1 + x_2 \quad + 2x_4 &= 0 \\ x_3 - x_4 &= 0 \end{aligned}$$

Thus, x_1 & x_3 are pivots/dependent variables, while x_2 & x_4 are free. We therefore parametrize them:

let $\boxed{\begin{aligned} x_2 &= s \\ x_4 &= t \end{aligned}}$

and solve the above for x_1 & x_3:

$$x_1 + x_2 + 2x_4 = 0$$

$$x_3 - x_4 = 0$$

$$\Longrightarrow \boxed{\begin{aligned} x_1 &= -x_2 - 2x_4 = -s-2t \\ x_3 &= x_4 = t \end{aligned}}$$

Therefore, all sol'ns $\vec{x} = \langle x_1, x_2, x_3, x_4 \rangle$ are of the form

$$\boxed{\begin{pmatrix} x_1 \\ x_2 \\ x_3 \\ x_4 \end{pmatrix} = \begin{pmatrix} -s-2t \\ s \\ t \\ t \end{pmatrix} = s\begin{pmatrix} -1 \\ 1 \\ 0 \\ 0 \end{pmatrix} + t\begin{pmatrix} -2 \\ 0 \\ 1 \\ 1 \end{pmatrix}}$$

2. In the matrix A corresponding to the system $A\mathbf{x} = \mathbf{0}$ above, show that the pivot columns of A (the original A, not its row-reduced equivalent) are linearly independent.

therefore $\vec{a}_1 = \begin{pmatrix} 3 \\ 6 \end{pmatrix}$ & $\vec{a}_3 = \begin{pmatrix} 2 \\ 2 \end{pmatrix}$ are the original pivot cols.

$$A = \begin{pmatrix} 3 & 3 & 2 & 4 \\ 6 & 6 & 2 & 10 \end{pmatrix}$$

$$\Rightarrow \operatorname{rref}(A) = \begin{pmatrix} 1 & 1 & 0 & 2 \\ 0 & 0 & 1 & -1 \end{pmatrix}$$

pivot columns are 1 & 3

Let's show $\vec{a}_1$ & $\vec{a}_3$ are linearly indep.

Suppose

$$a\vec{a}_1 + b\vec{a}_3 = \vec{0}$$

i.e. $$a\begin{pmatrix} 3 \\ 6 \end{pmatrix} + b\begin{pmatrix} 2 \\ 2 \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \end{pmatrix}$$

then $$\begin{pmatrix} 3 & 2 \\ 6 & 2 \end{pmatrix}\begin{pmatrix} a \\ b \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \end{pmatrix}$$

The matrix $\begin{pmatrix} 3 & 2 \\ 6 & 2 \end{pmatrix}$ row-reduces to $\begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}$:

$$\begin{pmatrix} 3 & 2 \\ 6 & 2 \end{pmatrix} \xrightarrow{-2\mathrm{I}+\mathrm{II}} \begin{pmatrix} 3 & 2 \\ 0 & -2 \end{pmatrix} \xrightarrow{\mathrm{II}+\mathrm{I}} \begin{pmatrix} 3 & 0 \\ 0 & 2 \end{pmatrix} \xrightarrow[\frac{1}{2}\mathrm{II}]{\frac{1}{3}\mathrm{I}} \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}$$

so $a = b = 0$.