Quiz 4

1. Consider two vectors $\mathbf{u}, \mathbf{v} \in \mathbb{R}^n$. If $\mathbf{u}$ is not scalar multiple of $\mathbf{v}$, must the pair be linearly independent? Prove your result by using the definition of linear independence.

Suppose $\vec{u} \notin \text{span}(\vec{v})$ & $\vec{v} \notin \text{span}(\vec{u})$. Then, if $\vec{u}$ & $\vec{v}$ were linearly dependent, there would be scalars $a, b \in \mathbb{R}$, not both 0, such that $a\vec{u} + b\vec{v} = \vec{0}$. Say $a \neq 0$, then $\vec{u} = -\frac{b}{a}\vec{v} \in \text{span}(\vec{v})$, a contradiction. Similarly, if $b \neq 0$, then $\vec{v} = \frac{-a}{b}\vec{u} \in \text{span}(\vec{u})$, a contradiction. Thus, $a = b = 0$ must hold.

2. Determine whether the vectors $\mathbf{u} = \begin{pmatrix} 1 \\ 1 \\ -2 \end{pmatrix}$, $\mathbf{v} = \begin{pmatrix} 5 \\ 1 \\ 2 \end{pmatrix}$, $\mathbf{w} = \begin{pmatrix} 7 \\ 0 \\ -5 \end{pmatrix}$ are linearly independent.

Let $a, b, c \in \mathbb{R}$ satisfy

$$\vec{0} = a\vec{u} + b\vec{v} + c\vec{w}$$
$$= a\begin{pmatrix} 1 \\ 1 \\ -2 \end{pmatrix} + b\begin{pmatrix} 5 \\ 1 \\ 2 \end{pmatrix} + c\begin{pmatrix} 7 \\ 0 \\ -5 \end{pmatrix}$$
$$= \begin{pmatrix} 1 & 5 & 7 \\ 1 & 1 & 0 \\ -2 & 2 & -5 \end{pmatrix}\begin{pmatrix} a \\ b \\ c \end{pmatrix} = A\vec{x},$$

$A = \begin{pmatrix} 1 & 5 & 7 \\ 1 & 1 & 0 \\ -2 & 2 & -5 \end{pmatrix}$, $\vec{x} = \begin{pmatrix} a \\ b \\ c \end{pmatrix} \in \mathbb{R}^3$. Thus, to solve

$\vec{0}$, row-reduce A: $\text{rref}\, A = \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix}$, which

$\vec{x} = \vec{0}$, or $a = b = c = 0$. Yes, $\{\vec{u}, \vec{v}, \vec{w}\}$ is lin. indep.