Quiz 17

(1) Determine which of the following matrices are diagonalizable. You do not need to actually diagonalize them, but do explain how you know whether or not they are diagonalizable.

(a) $A = \begin{pmatrix} 1 & 0 \\ 2 & 3 \end{pmatrix}$

$$\det(A - \lambda I) = \det\begin{pmatrix} 1-\lambda & 0 \\ 2 & 3-\lambda \end{pmatrix}$$

$$= (1-\lambda)(3-\lambda) = 0 \implies \lambda_1 = 1, \quad \lambda_2 = 3$$

2 distinct eigenvalues for $A \in M_2(\mathbb{R}) \implies$ $\underline{A \text{ is diagonalizable}}$

(b) $A = \begin{pmatrix} 1 & 2 \\ 3 & 4 \end{pmatrix}$

$$\det(A - \lambda I) = \det\begin{pmatrix} 1-\lambda & 2 \\ 3 & 4-\lambda \end{pmatrix} = (1-\lambda)(4-\lambda) - 6$$

$$= \lambda^2 - 5\lambda - 2$$

$$= 0$$

$$\implies \underbrace{\lambda = \frac{5 \pm \sqrt{25+8}}{2}}_{\text{2 distinct}} \implies \underline{\text{diagonalizable}}$$

(c) $A = \begin{pmatrix} 1 & 1 \\ -1 & 3 \end{pmatrix}$

$$\det(A - \lambda I) = \det\begin{pmatrix} 1-\lambda & 1 \\ -1 & 3-\lambda \end{pmatrix} = (1-\lambda)(3-\lambda) + 1$$

$$= \lambda^2 - 4\lambda + 4$$

$$= 0$$

$$\implies \lambda = \frac{4 \pm \sqrt{16-16}}{2} = 2$$ $\underline{\text{one eigenvalue}}$ $\underline{\text{w/ alg. mult. 2}}$

Check geom. mult.:

$\dim E_\lambda = \dim N(A - 2I)$

$= \dim N\begin{pmatrix} -1 & 1 \\ -1 & 1 \end{pmatrix}$

$= \dim N\begin{pmatrix} 1 & -1 \\ 0 & 0 \end{pmatrix} = 1 < 2$

$\underline{\text{not diagonalizable}}$

next

(2) Find a suitable basis $\rho = (\mathbf{p}_1, \mathbf{p}_2)$ of eigenvectors for the matrix $A = \begin{pmatrix} 2 & -6 \\ -1 & 3 \end{pmatrix}$ such that $[A]_\rho$ is diagonal.

$$\det(A - \lambda I) = \det\begin{pmatrix} 2-\lambda & -6 \\ -1 & 3-\lambda \end{pmatrix}$$

$$= (2-\lambda)(3-\lambda) - 6$$

$$= \lambda^2 - 5\lambda$$

$$= \lambda(\lambda - 5)$$

$$\Longrightarrow \lambda = 0, 5$$

$\underline{\lambda_1 = 0}$: $\quad N(A) = N\begin{pmatrix} 1 & -3 \\ 0 & 0 \end{pmatrix} = \operatorname{span}\underbrace{\begin{pmatrix} 3 \\ 1 \end{pmatrix}}_{\vec{v}_1}$

$\underline{\lambda_2 = 5}$: $\quad N(A - 5I) = N\begin{pmatrix} -3 & -6 \\ -1 & -2 \end{pmatrix}$

$$= N\begin{pmatrix} 1 & 2 \\ 0 & 0 \end{pmatrix}$$

$$= \operatorname{span}\underbrace{\begin{pmatrix} 2 \\ -1 \end{pmatrix}}_{v_2}$$

$$\Longrightarrow \quad Av_1 = \begin{pmatrix} 2 & -6 \\ -1 & 3 \end{pmatrix}\begin{pmatrix} 3 \\ 1 \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \end{pmatrix} = 0\begin{pmatrix} 3 \\ 1 \end{pmatrix} = 0\vec{v}_1 \checkmark$$

$$A\vec{v}_2 = \begin{pmatrix} 2 & -6 \\ -1 & 3 \end{pmatrix}\begin{pmatrix} 2 \\ -1 \end{pmatrix} = \begin{pmatrix} 10 \\ -5 \end{pmatrix} = 5\begin{pmatrix} 2 \\ -1 \end{pmatrix} = 5\vec{v}_2 \checkmark$$