Quiz 16

Determine whether the following operators and matrices are diagonalizable. If so, find a diagonalizing basis of eigenvectors.

(1) $A = \begin{pmatrix} 0 & -2 \\ 1 & 3 \end{pmatrix}$ in $M_2(\mathbb{R})$.

$$c_A(\lambda) = \det(A - \lambda I)$$
$$= \det\begin{pmatrix} -\lambda & -2 \\ 1 & 3-\lambda \end{pmatrix}$$
$$= -\lambda(3-\lambda) + 2$$
$$= \lambda^2 - 3\lambda + 2$$
$$= (\lambda - 1)(\lambda - 2)$$
$$= 0$$

$\Longrightarrow \boxed{\lambda_1 = 1,\ \lambda_2 = 2 \text{ eigenvalues}}$ (each with algebraic multiplicity $= 1$)

$\underline{\lambda_1 = 1}$: $A - \lambda_1 I = \begin{pmatrix} -1 & -2 \\ 1 & 2 \end{pmatrix} \longrightarrow \begin{pmatrix} 1 & 2 \\ 0 & 0 \end{pmatrix} = \mathrm{rref}(A - \lambda_1 I)$

Letting $y = t$, $x = -2t$, so

$\begin{pmatrix} x \\ y \end{pmatrix} = t\begin{pmatrix} -2 \\ 1 \end{pmatrix}$, i.e.

$\boxed{\vec{v}_1 = \begin{pmatrix} -2 \\ 1 \end{pmatrix}}$ eigenvector, & basis for $E_{\lambda_1} = E_1$ (hence, $\dim E_{\lambda_1} = \dim N(A - \lambda_1 I) = 1$)

$\underline{\lambda_2 = 2}$: $\quad A - \lambda_2 I = \begin{pmatrix} -2 & -2 \\ 1 & 1 \end{pmatrix} \longrightarrow \begin{pmatrix} 1 & 1 \\ 0 & 0 \end{pmatrix} = \text{rref}(A - \lambda_2 I)$

Let $y = t$, then $x = -t$, so

$$\begin{pmatrix} x \\ y \end{pmatrix} = t \begin{pmatrix} -1 \\ 1 \end{pmatrix}$$

$\Longrightarrow \boxed{\vec{v}_2 = \begin{pmatrix} -1 \\ 1 \end{pmatrix}}$ basis eigenvector for $E_{\lambda_2} = E_2$, which has dimension 1 ✓

$\underline{\text{Verify}}$: $\quad A\vec{v}_1 = \begin{pmatrix} 0 & -2 \\ 1 & 3 \end{pmatrix} \begin{pmatrix} -2 \\ 1 \end{pmatrix} = \begin{pmatrix} -2 \\ 1 \end{pmatrix} = 1 \cdot \vec{v}_1$ ✓

$= \lambda_1 \vec{v}_1$

& $\quad A\vec{v}_2 = \begin{pmatrix} 0 & -2 \\ 1 & 3 \end{pmatrix} \begin{pmatrix} -1 \\ 1 \end{pmatrix} = \begin{pmatrix} -2 \\ 2 \end{pmatrix} = 2 \begin{pmatrix} -1 \\ 1 \end{pmatrix} = 2\vec{v}_2$ ✓

$= \lambda_2 \vec{v}_2$

Hence, $\beta = (\vec{v}_1, \vec{v}_2) = \left(\begin{pmatrix} -2 \\ 1 \end{pmatrix}, \begin{pmatrix} -1 \\ 1 \end{pmatrix} \right)$ is a diagonalizing basis: $\quad M_{\beta\sigma} = \begin{pmatrix} | & | \\ \vec{v}_1 & \vec{v}_2 \\ | & | \end{pmatrix} = \begin{pmatrix} -2 & -1 \\ 1 & 1 \end{pmatrix}$, $\quad M_{\sigma\beta} = M_{\beta\sigma}^{-1} = \frac{-1}{1} \begin{pmatrix} 1 & 1 \\ -1 & -2 \end{pmatrix}$

$= \begin{pmatrix} -1 & -1 \\ 1 & 2 \end{pmatrix}$

$\Longrightarrow \boxed{M_{\beta\sigma}^{-1} A M_{\beta\sigma} = \begin{pmatrix} -1 & -1 \\ 1 & 2 \end{pmatrix} \begin{pmatrix} 0 & -2 \\ 1 & 3 \end{pmatrix} \begin{pmatrix} -2 & -1 \\ 1 & 1 \end{pmatrix} = \begin{pmatrix} 1 & 0 \\ 0 & 2 \end{pmatrix}}$

(2) $T \in \mathcal{L}(\mathbb{P}_2)$ given by $T(f) = f(0) + f(1)(x + x^2)$.

Let's use the "standard" basis for $\mathbb{P}_2$,

$$\sigma = (1, x, x^2)$$

& apply T to each vector:

$$T(1) = 1 + 1(x+x^2) = 1 + x + x^2 \Rightarrow [T(1)]_\sigma = \begin{pmatrix} 1 \\ 1 \\ 1 \end{pmatrix}$$

$$T(x) = 0 + 1(x+x^2) = 0\cdot 1 + x + x^2 \Rightarrow [T(x)]_\sigma = \begin{pmatrix} 0 \\ 1 \\ 1 \end{pmatrix}$$

$$T(x^2) = 0 + 1(x+x^2) = 0\cdot 1 + x + x^2 \Rightarrow [T(x^2)]_\sigma = \begin{pmatrix} 0 \\ 1 \\ 1 \end{pmatrix}$$

$$\Rightarrow [T]_\sigma = \left([T(1)]_\sigma \; [T(x)]_\sigma \; [T(x^2)]_\sigma \right)$$

$$= \begin{pmatrix} 1 & 0 & 0 \\ 1 & 1 & 1 \\ 1 & 1 & 1 \end{pmatrix}$$

$$\Rightarrow c_T(\lambda) = \det([T]_\sigma - \lambda I_3)$$

$$= \det \begin{pmatrix} 1-\lambda & 0 & 0 \\ 1 & 1-\lambda & 1 \\ 1 & 1 & 1-\lambda \end{pmatrix}$$

$$= (1-\lambda) \det \begin{pmatrix} 1-\lambda & 1 \\ 1 & 1-\lambda \end{pmatrix} + 0 + 0$$

$$= (1-\lambda)((1-\lambda)^2 - 1)$$

$$= (1-\lambda)(\lambda^2 - 2\lambda) = \lambda(1-\lambda)(\lambda-2)$$

$\Rightarrow \boxed{\lambda = 0, 1, 2}$ eigenvalues (each with algebraic)

Now for the eigenvectors:

$\underline{\lambda_1 = 0}$: $A - 0I = A = \begin{pmatrix} 1 & 0 & 0 \\ 1 & 1 & 1 \\ 1 & 1 & 1 \end{pmatrix}$

$(A = [T]_\sigma)$ $\longrightarrow \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 1 \\ 0 & 0 & 0 \end{pmatrix} = \text{rref}\, A$

Let $z = t$, then $y = -t$, $x = 0$, so

$$\begin{pmatrix} x \\ y \\ z \end{pmatrix} = \begin{pmatrix} 0 \\ -t \\ t \end{pmatrix} = t \begin{pmatrix} 0 \\ -1 \\ 1 \end{pmatrix}$$

$\Rightarrow \vec{v}_1 = \begin{pmatrix} 0 \\ -1 \\ 1 \end{pmatrix}$ basis eigenvector for $E_{\lambda_1} = E_0$ in $\mathbb{R}^3$

$$\boxed{p_1(x) = -x + x^2}$$

$\underline{\lambda_2 = 1}$: $A - 1I_3 = \begin{pmatrix} 0 & 0 & 0 \\ 1 & 0 & 1 \\ 1 & 1 & 0 \end{pmatrix} \longrightarrow \begin{pmatrix} 1 & 0 & 1 \\ 0 & 1 & -1 \\ 0 & 0 & 0 \end{pmatrix} = \text{rref}(A - I)$

Let $z = t$, then $y = t$, $x = -t$

$\Rightarrow \begin{pmatrix} x \\ y \\ z \end{pmatrix} = t \begin{pmatrix} -1 \\ 1 \\ 1 \end{pmatrix}$

~~Let $z = t$, then $x = -t$, $y = 0$ $\Rightarrow$ $\begin{pmatrix} x \\ y \\ z \end{pmatrix} = t\begin{pmatrix} -1 \\ 0 \\ 1 \end{pmatrix} \Rightarrow -1 + x^2 = p_2$ basis eigenvector for $E_{\lambda_2} = E_1$~~

$\Rightarrow \boxed{p_2(x) = -1 + x + x^2 \text{ basis eigenvector for } E_{\lambda_2} = E_1}$

$\underline{\lambda_3 = 2:}$ $\quad A - \lambda_3 I = \begin{pmatrix} 1 & 0 & 0 \\ 1 & 1 & 1 \\ 1 & 1 & 1 \end{pmatrix} - \begin{pmatrix} 2 & 0 & 0 \\ 0 & 2 & 0 \\ 0 & 0 & 2 \end{pmatrix}$

$$= \begin{pmatrix} -1 & 0 & 0 \\ 1 & -1 & 1 \\ 1 & 1 & -1 \end{pmatrix}$$

$$\longrightarrow \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & -1 \\ 0 & 0 & 0 \end{pmatrix} = \mathrm{rref}(A - \lambda_3 I)$$

Let $z = t$, then $y = t$, $x = 0$ $\Longrightarrow \begin{pmatrix} x \\ y \\ z \end{pmatrix} = t \begin{pmatrix} 0 \\ 1 \\ 1 \end{pmatrix}$

$\Longrightarrow$ $\boxed{\vec{p}_3(x) = x + x^2 \text{ basis eigenvector for } E_{\lambda_3} = E_2}$

$\underline{\text{Verify:}}$ $\quad T(\vec{p}_1(x)) = \vec{p}_1(0) + p_1(1)(x + x^2)$

$$= (-0 + 0^2) + (-1 + 1^2)(x + x^2)$$

$$= 0 + 0$$

$$= 0$$

$$= 0\vec{p}_1(x) \quad \checkmark$$

$$T(p_2(x)) = p_2(0) + p_1(1)(x+x^2)$$

$$= (\overset{-1}{0}+0^2) + (\overset{-1}{1}+1^2)(x+x^2) = -1+x+x^2$$

$$= 1 \cdot p_1(x) \checkmark$$

$$T(p_3(x)) = p_3(0) + p_1(1)(x+x^2)$$

$$= (0+0^2) + (1+1^2)(x+x^2)$$

$$= 2(x+x^2)$$

$$= 2p_3(x) \checkmark$$

Therefore,

$$\boxed{\begin{aligned} \beta &= (p_1(x), p_2(x), p_3(x)) \\ &= (-x+x^2,\ -1+x+x^2,\ x+x^2) \end{aligned}}$$

is a diagonalizing basis for T, &

$$\boxed{[T]_\beta = \begin{pmatrix} 0 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 2 \end{pmatrix}}$$