Quiz 15

(1) Let R_ℓ be the reflection in $\mathbb{R}^2$ across the line

$$\ell = \left\{ \mathbf{x} = \begin{pmatrix} x \\ y \end{pmatrix} \mid 2x + y = 0 \right\}$$

Find a natural basis β for R_ℓ, in which its representation is

$$[R_\ell]_\beta = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}$$

and use this to find the standard representation $[R_\ell]_\sigma$. (Remember, you want $\beta = (\mathbf{b}_1, \mathbf{b}_2)$ with $R_\ell(\mathbf{b}_1) = \mathbf{b}_1$ and $R_\ell(\mathbf{b}_2) = -\mathbf{b}_2$.)

$\vec{b}_1 = \begin{pmatrix} 1 \\ -2 \end{pmatrix}$ by letting $x = 1,\ y = -2$

$\vec{b}_2 = R_{\pi/2}\vec{b}_1 = \begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix}\begin{pmatrix} 1 \\ -2 \end{pmatrix} = \begin{pmatrix} 2 \\ 1 \end{pmatrix}$

$\Rightarrow\ M_{\beta\sigma} = (\vec{b}_1\ \vec{b}_2) = \begin{pmatrix} 1 & 2 \\ -2 & 1 \end{pmatrix}$

$M_{\sigma\beta} = M_{\beta\sigma}^{-1} = \frac{1}{5}\begin{pmatrix} 1 & -2 \\ 2 & 1 \end{pmatrix}$

$$\Rightarrow\ [R_\ell]_\sigma = M_{\beta\sigma}[R_\ell]\overbrace{M_{\beta\sigma}^{-1}}^{=M_{\sigma\beta}}$$

$$= \begin{pmatrix} 1 & 2 \\ -2 & 1 \end{pmatrix}\begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}\left(\frac{1}{5}\begin{pmatrix} 1 & -2 \\ 2 & 1 \end{pmatrix}\right)$$

$$= \frac{1}{5}\begin{pmatrix} 1 & -2 \\ -2 & -1 \end{pmatrix}\begin{pmatrix} 1 & -2 \\ 2 & 1 \end{pmatrix}$$

$$= \frac{1}{5}\begin{pmatrix} -3 & -4 \\ -4 & 3 \end{pmatrix}$$

(2) Use projections onto the β basis vectors you got in (1) to decompose the vector $\mathbf{x} = \begin{pmatrix}1\\1\end{pmatrix}$ into parallel and orthogonal components to $\ell = \text{span}(\mathbf{b}_1)$.

$$\vec{x}_{\parallel} = \pi_{\vec{b}_1}\vec{x} = \frac{\vec{b}_1 \cdot \vec{x}}{\|\vec{b}_1\|^2}\vec{b}_1 = \frac{\begin{pmatrix}1\\-2\end{pmatrix}\cdot\begin{pmatrix}1\\1\end{pmatrix}}{1^2+(-2)^2}\begin{pmatrix}1\\-2\end{pmatrix}$$

$$= \boxed{\frac{-1}{5}\begin{pmatrix}1\\-2\end{pmatrix}}$$

$$\vec{x}_{\perp} = \pi_{\vec{b}_2}\vec{x} = \frac{\vec{b}_2 \cdot \vec{x}}{\|\vec{b}_2\|^2}\vec{b}_2 = \frac{\begin{pmatrix}2\\1\end{pmatrix}\cdot\begin{pmatrix}1\\1\end{pmatrix}}{\left\|\begin{pmatrix}2\\1\end{pmatrix}\right\|^2}\begin{pmatrix}2\\1\end{pmatrix}$$

$$= \boxed{\frac{3}{5}\begin{pmatrix}2\\1\end{pmatrix}}$$

Note: $$\vec{x}_{\parallel} + \vec{x}_{\perp} = \frac{-1}{5}\begin{pmatrix}1\\-2\end{pmatrix} + \frac{3}{5}\begin{pmatrix}2\\1\end{pmatrix}$$

$$= \begin{pmatrix}-1/5 + 6/5\\ 2/5 + 3/5\end{pmatrix}$$

$$= \begin{pmatrix}1\\1\end{pmatrix} = \vec{x} \quad \checkmark$$