

Quiz 14

Consider the standard and a non-standard basis for \mathbb{R}^2 ,

$$\sigma = (\mathbf{e}_1, \mathbf{e}_2) = \left(\begin{pmatrix} 1 \\ 0 \end{pmatrix}, \begin{pmatrix} 0 \\ 1 \end{pmatrix} \right), \quad \beta = (\mathbf{b}_1, \mathbf{b}_2) = \left(\begin{pmatrix} 1 \\ 2 \end{pmatrix}, \begin{pmatrix} 3 \\ 1 \end{pmatrix} \right),$$

and define the β -coordinate representation of a vector $\mathbf{x} = \begin{pmatrix} 3 \\ 1 \end{pmatrix} \in \mathbb{R}^2$ to be

$$[\mathbf{x}]_\beta = \begin{pmatrix} a \\ b \end{pmatrix}, \quad \text{whenever } \mathbf{x} = a\mathbf{b}_1 + b\mathbf{b}_2$$

and the β -coordinate representation of a matrix $A = \begin{pmatrix} 1 & -2 \\ -3 & 4 \end{pmatrix} \in M_2(\mathbb{R})$ to be

$$[A]_\beta = ([A\mathbf{b}_1]_\beta \quad [A\mathbf{b}_2]_\beta)$$

The change-of-coordinates matrix changing β -coordinates to γ -coordinates is

$$M_{\beta, \gamma} = \left([\mathbf{b}_1]_\gamma \quad [\mathbf{b}_2]_\gamma \right)$$

You may use the following facts in your calculations:

- $[\mathbf{x}]_\sigma = \mathbf{x}$ for all $\mathbf{x} \in \mathbb{R}^2$
- $[\mathbf{x}]_\gamma = M_{\beta, \gamma} [\mathbf{x}]_\beta$
- $M_{\beta, \sigma} = ([\mathbf{b}_1]_\sigma \quad [\mathbf{b}_2]_\sigma) = (\mathbf{b}_1 \quad \mathbf{b}_2)$
- $M_{\gamma, \beta} = M_{\beta, \gamma}^{-1}$
- $M_{\beta, \delta} = M_{\gamma, \delta} M_{\beta, \gamma}$ for any three bases β, γ, δ .

1. Find $[\mathbf{x}]_\beta$.

$$\begin{aligned} [\vec{x}]_\beta &= M_{\sigma\beta}^{-1} [\vec{x}]_\sigma = M_{\beta\sigma}^{-1} \vec{x} = \begin{pmatrix} 1 & 3 \\ 2 & 1 \end{pmatrix}^{-1} \begin{pmatrix} 3 \\ 1 \end{pmatrix} \\ &= -\frac{1}{5} \begin{pmatrix} 1 & -3 \\ -2 & 1 \end{pmatrix} \begin{pmatrix} 3 \\ 1 \end{pmatrix} = -\frac{1}{5} \begin{pmatrix} 0 \\ -5 \end{pmatrix} \end{aligned}$$

2. Find $[A]_\beta$.

$$\begin{aligned} [A]_\beta &= M_{\sigma\beta} [A]_\sigma M_{\beta\sigma} \\ &= -\frac{1}{5} \begin{pmatrix} 1 & -3 \\ -2 & 1 \end{pmatrix} \begin{pmatrix} 1 & -2 \\ -3 & 4 \end{pmatrix} \begin{pmatrix} 1 & 3 \\ 2 & 1 \end{pmatrix} = -\frac{1}{5} \begin{pmatrix} 10 & -14 \\ -5 & 8 \end{pmatrix} \begin{pmatrix} 1 & 3 \\ 2 & 1 \end{pmatrix} \end{aligned}$$

$$= -\frac{1}{5} \begin{pmatrix} -18 & 16 \\ 11 & -7 \end{pmatrix}$$