Quiz 13

Consider the standard and two non-standard bases for $\mathbb{R}^2$,

$$\sigma = (\mathbf{e}_1, \mathbf{e}_2) = \left(\begin{pmatrix}1\\0\end{pmatrix}, \begin{pmatrix}0\\1\end{pmatrix}\right), \quad \beta = (\mathbf{b}_1, \mathbf{b}_2) = \left(\begin{pmatrix}1\\-2\end{pmatrix}, \begin{pmatrix}0\\1\end{pmatrix}\right), \quad \gamma = (\mathbf{c}_1, \mathbf{c}_2) = \left(\begin{pmatrix}1\\1\end{pmatrix}, \begin{pmatrix}2\\3\end{pmatrix}\right)$$

and define the β-coordinate representation of a vector $\mathbf{x} \in \mathbb{R}^2$ to be

$$[\mathbf{x}]_\beta = \begin{pmatrix}a\\b\end{pmatrix}, \quad \text{whenever} \quad \mathbf{x} = a\mathbf{b}_1 + b\mathbf{b}_2$$

We similarly define $[\mathbf{x}]_\gamma$. The change-of-coordinates matrix changing β-coordinates to γ-coordinates is

$$M_{\beta,\gamma} = \begin{pmatrix}[\mathbf{b}_1]_\gamma & [\mathbf{b}_2]_\gamma\end{pmatrix}$$

You may use the following facts in your calculations:

- $[\mathbf{x}]_\sigma = \mathbf{x}$ for all $\mathbf{x} \in \mathbb{R}^2$
- $[\mathbf{x}]_\gamma = M_{\beta,\gamma}[\mathbf{x}]_\beta$
- $M_{\beta,\sigma} = \left([\mathbf{b}_1]_\sigma \ [\mathbf{b}_2]_\sigma\right) = (\mathbf{b}_1 \ \mathbf{b}_2)$
- $M_{\gamma,\beta} = M_{\beta,\gamma}^{-1}$
- $M_{\beta,\delta} = M_{\gamma,\delta}M_{\beta,\gamma}$ for any three bases β, γ, δ.

1. Find $M_{\beta,\sigma} = \left([\mathbf{b}_1]_\sigma \ [\mathbf{b}_2]_\sigma\right)$ and $M_{\sigma,\beta} = \left([\mathbf{e}_1]_\beta \ [\mathbf{e}_2]_\beta\right)$.

$$M_{\beta\sigma} = (\vec{b}_1\ \vec{b}_2) = \begin{pmatrix}1 & 0\\-2 & 1\end{pmatrix}$$

$$M_{\sigma\beta} = M_{\beta\sigma}^{-1} = \begin{pmatrix}1 & 0\\2 & 1\end{pmatrix}$$

2. Find $M_{\gamma,\sigma} = \left([\mathbf{c}_1]_\sigma \ [\mathbf{c}_2]_\sigma\right)$ and $M_{\sigma,\gamma} = \left([\mathbf{e}_1]_\gamma \ [\mathbf{e}_2]_\gamma\right) = M_{\gamma,\sigma}^{-1}$.

$$M_{\gamma\sigma} = (\vec{c}_1\ \vec{c}_2) = \begin{pmatrix}1 & 2\\1 & 3\end{pmatrix}$$

$$M_{\sigma\gamma} = M_{\gamma\sigma}^{-1} = \begin{pmatrix}3 & -2\\-1 & 1\end{pmatrix}$$

3. Find $M_{\beta,\gamma} = ([\mathbf{b}_1]_\gamma \ \ [\mathbf{b}_2]_\gamma)$ and $M_{\gamma,\beta} = ([\mathbf{c}_1]_\beta \ \ [\mathbf{c}_2]_\beta) = M_{\beta,\gamma}^{-1}$.

$$M_{\beta\gamma} = M_{\sigma\gamma} M_{\beta\sigma} = \begin{pmatrix} 3 & -2 \\ -1 & 1 \end{pmatrix}\begin{pmatrix} 1 & 0 \\ -2 & 1 \end{pmatrix}$$

$$= \begin{pmatrix} 7 & -2 \\ -3 & 1 \end{pmatrix}$$

$$M_{\gamma\beta} = M_{\beta\gamma}^{-1} = \begin{pmatrix} 1 & 2 \\ 3 & 7 \end{pmatrix}$$

4. Let $\mathbf{x} = \begin{pmatrix} 5 \\ -3 \end{pmatrix}$. Use the above work to find $[\mathbf{x}]_\beta$ and $[\mathbf{x}]_\gamma$.

$$[\vec{x}]_\beta = M_{\sigma\beta}[\vec{x}]_\sigma = \begin{pmatrix} 1 & 0 \\ 2 & 1 \end{pmatrix}\begin{pmatrix} 5 \\ -3 \end{pmatrix} = \begin{pmatrix} 5 \\ 7 \end{pmatrix}$$

$$[\vec{x}]_\gamma = M_{\sigma\gamma}[\vec{x}]_\sigma = \begin{pmatrix} 3 & -2 \\ -1 & 1 \end{pmatrix}\begin{pmatrix} 5 \\ -3 \end{pmatrix} = \begin{pmatrix} 21 \\ -8 \end{pmatrix}$$

5. Verify that your representations for $\mathbf{x}$ in # 4 are correct: for example, if $[\mathbf{x}]_\beta = \binom{a}{b}$, then verify that $\mathbf{x} = a\mathbf{b}_1 + b\mathbf{b}_2$, and similarly with γ.

$$[\vec{x}]_\beta = \begin{pmatrix} 5 \\ 7 \end{pmatrix} \implies 5\vec{b}_1 + 7\vec{b}_2 = 5\begin{pmatrix} 1 \\ -2 \end{pmatrix} + 7\begin{pmatrix} 0 \\ 1 \end{pmatrix}$$

$$= \begin{pmatrix} 5+0 \\ -10+7 \end{pmatrix} = \begin{pmatrix} 5 \\ -3 \end{pmatrix} = \vec{x} \quad \checkmark$$

$$[\vec{x}]_\gamma = \begin{pmatrix} 21 \\ -8 \end{pmatrix} \implies \cancel{21\vec{b}_1 - 8\vec{b}_2 = 21\begin{pmatrix} 1 \\ -2 \end{pmatrix} - 8\begin{pmatrix} 0 \\ 1 \end{pmatrix} = \begin{pmatrix} 21+0 \\ -42-8 \end{pmatrix}}$$

$$21\vec{c}_1 - 8\vec{c}_2 = 21\begin{pmatrix} 1 \\ 1 \end{pmatrix} - 8\begin{pmatrix} 2 \\ 3 \end{pmatrix}$$

$$= \begin{pmatrix} 21-16 \\ 21-24 \end{pmatrix} = \begin{pmatrix} 5 \\ -3 \end{pmatrix} = \vec{x} \quad \checkmark$$