Quiz 13

Consider the standard and two non-standard bases for $\mathbb{R}^2$,

$$\sigma = (\mathbf{e}_1, \mathbf{e}_2) = \left(\begin{pmatrix} 1 \\ 0 \end{pmatrix}, \begin{pmatrix} 0 \\ 1 \end{pmatrix} \right), \quad \beta = (\mathbf{b}_1, \mathbf{b}_2) = \left(\begin{pmatrix} 1 \\ -2 \end{pmatrix}, \begin{pmatrix} 0 \\ 1 \end{pmatrix} \right), \quad \gamma = (\mathbf{c}_1, \mathbf{c}_2) = \left(\begin{pmatrix} 1 \\ 1 \end{pmatrix}, \begin{pmatrix} 2 \\ 3 \end{pmatrix} \right)$$

and define the β-coordinate representation of a vector $\mathbf{x} \in \mathbb{R}^2$ to be

$$[\mathbf{x}]_\beta = \begin{pmatrix} a \\ b \end{pmatrix}, \quad \text{whenever} \quad \mathbf{x} = a\mathbf{b}_1 + b\mathbf{b}_2$$

We similarly define $[\mathbf{x}]_\gamma$. The change-of-coordinates matrix changing β-coordinates to γ-coordinates is

$$M_{\beta,\gamma} = \begin{pmatrix} [\mathbf{b}_1]_\gamma & [\mathbf{b}_2]_\gamma \end{pmatrix}$$

You may use the following facts in your calculations:

- $[\mathbf{x}]_\sigma = \mathbf{x}$ for all $\mathbf{x} \in \mathbb{R}^2$
- $[\mathbf{x}]_\gamma = M_{\beta,\gamma}[\mathbf{x}]_\beta$
- $M_{\beta,\sigma} = \big([\mathbf{b}_1]_\sigma \;\; [\mathbf{b}_2]_\sigma\big) = \big(\mathbf{b}_1 \;\; \mathbf{b}_2\big)$
- $M_{\gamma,\beta} = M_{\beta,\gamma}^{-1}$
- $M_{\beta,\delta} = M_{\gamma,\delta} M_{\beta,\gamma}$ for any three bases β, γ, δ.

1. Find $M_{\beta,\sigma} = \big([\mathbf{b}_1]_\sigma \;\; [\mathbf{b}_2]_\sigma\big)$ and $M_{\sigma,\beta} = \big([\mathbf{e}_1]_\beta \;\; [\mathbf{e}_2]_\beta\big)$.

2. Find $M_{\gamma,\sigma} = \big([\mathbf{c}_1]_\sigma \;\; [\mathbf{c}_2]_\sigma\big)$ and $M_{\sigma,\gamma} = \big([\mathbf{e}_1]_\gamma \;\; [\mathbf{e}_2]_\gamma\big) = M_{\gamma,\sigma}^{-1}$.

3. Find $M_{\beta,\gamma} = \left([\mathbf{b}_1]_\gamma \;\; [\mathbf{b}_2]_\gamma\right)$ and $M_{\gamma,\beta} = \left([\mathbf{c}_1]_\beta \;\; [\mathbf{c}_2]_\beta\right) = M_{\beta,\gamma}^{-1}$.

4. Let $\mathbf{x} = \begin{pmatrix} 5 \\ -3 \end{pmatrix}$. Use the above work to find $[\mathbf{x}]_\beta$ and $[\mathbf{x}]_\gamma$.

5. Verify that your representations for $\mathbf{x}$ in # 4 are correct: for example, if $[\mathbf{x}]_\beta = \binom{a}{b}$, then verify that $\mathbf{x} = a\mathbf{b}_1 + b\mathbf{b}_2$, and similarly with γ.