Quiz 12

Consider the standard and two non-standard bases for $\mathbb{R}^2$,

$$\sigma = (\mathbf{e}_1, \mathbf{e}_2) = \left(\begin{pmatrix} 1 \\ 0 \end{pmatrix}, \begin{pmatrix} 0 \\ 1 \end{pmatrix} \right), \quad \beta = (\mathbf{b}_1, \mathbf{b}_2) = \left(\begin{pmatrix} 2 \\ 1 \end{pmatrix}, \begin{pmatrix} -1 \\ 1 \end{pmatrix} \right), \quad \gamma = (\mathbf{c}_1, \mathbf{c}_2) = \left(\begin{pmatrix} 0 \\ 1 \end{pmatrix}, \begin{pmatrix} 1 \\ 1 \end{pmatrix} \right)$$

and define the β-coordinate representation of a vector $\mathbf{x} \in \mathbb{R}^2$ to be

$$[\mathbf{x}]_\beta = \begin{pmatrix} a \\ b \end{pmatrix} \quad \text{means} \quad \mathbf{x} = a\mathbf{b}_1 + b\mathbf{b}_2$$

and similarly for $[\mathbf{x}]_\gamma$ (of course, $[\mathbf{x}]_\sigma = \mathbf{x}$).

1. Find $M_{\beta,\sigma} = ([\mathbf{b}_1]_\sigma, [\mathbf{b}_2]_\sigma)$ and $M_{\sigma,\beta} = ([\mathbf{e}_1]_\beta, [\mathbf{e}_2]_\beta) = M_{\beta,\sigma}^{-1}$.

$$M_{\beta\sigma} = (\vec{b}_1\ \vec{b}_2) = \begin{pmatrix} 2 & -1 \\ 1 & 1 \end{pmatrix}$$

$$M_{\sigma\beta} = M_{\beta\sigma}^{-1} = \begin{pmatrix} 2 & -1 \\ 1 & 1 \end{pmatrix}^{-1}$$

$$= \frac{1}{1} \begin{pmatrix} 1 & 1 \\ -1 & 2 \end{pmatrix}$$

$$= \begin{pmatrix} 1 & 1 \\ -1 & 2 \end{pmatrix}$$

2. Find $M_{\gamma,\sigma} = ([\mathbf{c}_1]_\sigma, [\mathbf{c}_2]_\sigma)$ and $M_{\sigma,\gamma} = ([\mathbf{e}_1]_\gamma, [\mathbf{e}_2]_\gamma) = M_{\gamma,\sigma}^{-1}$.

$$M_{\gamma\sigma} = (\vec{c}_1\ \vec{c}_2) = \begin{pmatrix} 0 & 1 \\ 1 & 1 \end{pmatrix}$$

$$M_{\sigma\gamma} = M_{\gamma\sigma}^{-1} = \begin{pmatrix} 0 & 1 \\ 1 & 1 \end{pmatrix}^{-1}$$

$$= \frac{1}{-1} \begin{pmatrix} 1 & -1 \\ -1 & 0 \end{pmatrix}$$

$$= \begin{pmatrix} -1 & 1 \\ 1 & 0 \end{pmatrix}$$

3. Find $M_{\beta,\gamma} = \big([\mathbf{b}_1]_\gamma, [\mathbf{b}_2]_\gamma\big)$ and $M_{\gamma,\beta} = \big([\mathbf{c}_1]_\beta, [\mathbf{c}_2]_\beta\big) = M_{\beta,\gamma}^{-1}$.

$$M_{\beta\gamma} = M_{\sigma\gamma} M_{\beta\sigma}$$
$$= \begin{pmatrix} -1 & 1 \\ 1 & 0 \end{pmatrix}\begin{pmatrix} 2 & -1 \\ 1 & 1 \end{pmatrix}$$
$$= \begin{pmatrix} -1 & 2 \\ 2 & -1 \end{pmatrix}$$

$$M_{\gamma\beta} = M_{\beta\gamma}^{-1} = \begin{pmatrix} -1 & 2 \\ 2 & -1 \end{pmatrix}^{-1} = -\frac{1}{3}\begin{pmatrix} -1 & -2 \\ -2 & -1 \end{pmatrix}$$
$$= \frac{1}{3}\begin{pmatrix} 1 & 2 \\ 2 & 1 \end{pmatrix}$$

4. Verify that the components of the vector $[\mathbf{c}_1]_\beta$ are the coefficients c_i of $\mathbf{c}_1 = c_1\mathbf{b}_1 + c_2\mathbf{b}_2$ and the components of the vector $[\mathbf{c}_2]_\beta$ are the coefficients d_i of $\mathbf{c}_2 = d_1\mathbf{b}_1 + d_2\mathbf{b}_2$.

$$[\vec{c}_1]_\beta = \frac{1}{3}\begin{pmatrix} 1 \\ 2 \end{pmatrix} \;\&\; [\vec{c}_2]_\beta = \frac{1}{3}\begin{pmatrix} 2 \\ 1 \end{pmatrix}$$

$$\Rightarrow \frac{1}{3}\vec{b}_1 + \frac{2}{3}\vec{b}_2 = \frac{1}{3}\begin{pmatrix} 2 \\ 1 \end{pmatrix} + \frac{2}{3}\begin{pmatrix} -1 \\ 1 \end{pmatrix}$$
$$= \begin{pmatrix} 2/3 - 2/3 \\ 1/3 + 2/3 \end{pmatrix}$$
$$= \begin{pmatrix} 0 \\ 1 \end{pmatrix} = \vec{c}_1 \checkmark$$

$$\& \; \frac{2}{3}\vec{b}_1 + \frac{1}{3}\vec{b}_2 = \frac{2}{3}\begin{pmatrix} 2 \\ 1 \end{pmatrix} + \frac{1}{3}\begin{pmatrix} -1 \\ 1 \end{pmatrix}$$
$$= \begin{pmatrix} 4/3 - 1/3 \\ 2/3 + 1/3 \end{pmatrix} = \begin{pmatrix} 1 \\ 1 \end{pmatrix} = \vec{c}_2 \checkmark$$