HW 4

Chapter 4 Problems

(41) $P(X \geq 7 \mid F^c) = \boxed{\dfrac{\binom{10}{7}}{2^{10}} + \dfrac{\binom{10}{8}}{2^{10}} + \dfrac{\binom{10}{9}}{2^{10}} + \dfrac{\binom{10}{10}}{2^{10}}}$

$= P(\text{equally likely})$

(58) (a) $P(X=2) = \boxed{\binom{8}{2} p^2 q^6} = 28 \cdot \frac{1}{10^2} \cdot \frac{9^6}{10^6} \approx 0.1488$ Binomial

$\begin{pmatrix} n = 8 \\ p = 0.1 \end{pmatrix}$

VS

$\begin{pmatrix} \lambda = np \\ = 8/10 \\ = 4/5 \end{pmatrix} \longrightarrow \approx \boxed{\frac{(4/5)^2}{2!} e^{-4/5} \approx 0.1438}$ Poisson

(b) $n = 10$, $p = 0.95 = 19/20$, $\lambda = np = 19/2$

Binomial

$\boxed{P(X=9) = \binom{10}{9}(0.95)^9(0.05)^1 \approx 0.3151}$

VS

$\approx \boxed{e^{-19/2} \cdot \frac{\left(\frac{19}{2}\right)^9}{9!} \approx 0.1300}$

Poisson

(c) $n=10,\ p=0.1,\ \lambda=np=1$

Binomial

$$\boxed{P(X=0)=\binom{10}{0}(0.1)^0(0.9)^{10} = (0.9)^{10} \approx 0.3487}$$

vs

Poisson

$$\boxed{\approx e^{-1}\cdot\frac{1^0}{0!} = e^{-1} \approx 0.3679}$$

(d) $n=9,\ p=0.2=1/5,\ np=9/5=1.8$

$$\boxed{P(X=4)=\binom{9}{4}(1/5)^4(4/5)^5 \approx 0.0661}$$

Binomial

vs

$$\boxed{\approx e^{-9/5}\cdot\frac{(9/5)^4}{4!} \approx 0.0723}$$

Poisson

(69) (a) $P(\text{string of 4 H's in 10})$

$$= \sum_{r=1}^{10-4+1} (-1)^{r+1} \left[\binom{10-4r}{r} + 2\binom{10-4r}{r-1} \right] \left(\tfrac{1}{2}\right)^{5r}$$

$$= (-1)^{1+1} \left[\binom{6}{1} + 2\binom{6}{0} \right] \left(\tfrac{1}{2}\right)^{5}$$

$r \geq 3$ makes no sense, so we count those as 0

$$+ (-1)^{2+1} \left[\binom{2}{2} + 2\binom{2}{1} \right] \left(\tfrac{1}{2}\right)^{10}$$

$$= \boxed{\frac{8}{2^5} - \frac{5}{2^{10}} = \frac{251}{1024} \approx 0.2451}$$

(b) $P_j \stackrel{\text{def}}{=}$ 4 consecutive heads happen in j flips

$$\Longrightarrow \quad P_1 = P_2 = P_3 = 0$$

$$P_4 = P(HHHH) = \frac{1}{2^4} = \frac{1}{16}$$

$$\Longrightarrow \quad P_5 = \left(\tfrac{1}{2}\right)^4 + \sum_{j=1}^{4} P_{5-j} \left(\tfrac{1}{2}\right)^j$$

$$= \frac{1}{16} + \frac{1}{2} P_4$$

$$= \frac{1}{16} + \frac{1}{32} = \frac{3}{32}$$

$$P_6 = \frac{1}{16} + \sum_{j=1}^{4} P_{6-j} \left(\frac{1}{2}\right)^j$$

$$= \frac{1}{16} + \frac{1}{2} P_5 + \frac{1}{2^2} P_4$$

$$= \frac{1}{16} + \frac{1}{2} \cdot \frac{3}{32} + \frac{1}{4} \cdot \frac{1}{16}$$

$$= \frac{4 + 3 + 1}{64} = \frac{8}{64} = \frac{1}{8}$$

$$P_7 = \frac{1}{16} + \frac{1}{2} P_6 + \frac{1}{4} P_5 + \frac{1}{8} P_4$$

$$= \frac{1}{16} + \frac{1}{2} \cdot \frac{1}{8} + \frac{1}{4} \cdot \frac{3}{32} + \frac{1}{8} \cdot \frac{1}{16}$$

$$= \frac{8 + 8 + 4}{128} = \frac{20}{128} = \frac{10}{64} = \frac{5}{32}$$

$$P_8 = \frac{1}{16} + \frac{1}{2} \cdot \frac{5}{32} + \frac{1}{4} \cdot \frac{1}{8} + \frac{1}{8} \cdot \frac{3}{32} + \frac{1}{16} \cdot \frac{1}{16}$$

$$= \frac{3}{16}$$

$$P_9 = \frac{1}{16} + \frac{1}{2} \cdot \frac{3}{16} + \frac{1}{4} \cdot \frac{5}{32} + \frac{1}{8} \cdot \frac{1}{8} + \frac{1}{16} \cdot \frac{3}{32} + \cancel{\frac{1}{32} \cdot \frac{1}{16}}$$

$$= \frac{111}{512}$$

$$\boxed{P_{10} =} \frac{1}{16} + \frac{1}{2} \cdot \frac{111}{512} + \frac{1}{4} \cdot \frac{3}{16} + \frac{1}{8} \cdot \frac{5}{32} + \frac{1}{16} \cdot \frac{1}{8} = \boxed{\frac{251}{1024}}$$

(c) $P(L_{10} \geq 4) \approx 1 - e^{-\{(10-4)(1/2)^5 + (1/2)^4\}}$

$$= 1 - e^{-(6/2^5 + 1/2^4)}$$

$$= 1 - e^{-8/2^5}$$

$$\boxed{\begin{aligned} &= 1 - e^{-1/2^2} \\ &\approx 0.2212 \end{aligned}}$$

Ch. 4 Theoretical Exercises:

(19) X Poisson $\Longrightarrow p(i) = P(X=i) = e^{-\lambda} \cdot \frac{\lambda^i}{i!}$

$$E[X^n] = \sum_{i=0}^{\infty} i^n p(i)$$

factor out a λ

$$= \sum_{i=0}^{\infty} i^{n-1} \cdot \cancel{i} \cdot e^{-\lambda} \cdot \frac{\lambda^i}{\cancel{i!}} \quad = (i-1)!$$

$$= \lambda \sum_{i=1}^{\infty} i^{n-1} \cdot e^{-\lambda} \cdot \frac{\lambda^{i-1}}{(i-1)!}$$

(let $j = i-1$ $\Longleftrightarrow i = j+1$)

$$= \lambda \underbrace{\sum_{j=0}^{\infty} (j+1)^{n-1} e^{-\lambda} \cdot \frac{\lambda^j}{j!}}_{= \sum_{j=0}^{\infty} (j+1)^{n-1} p(j) = E[(X+1)^{n-1}]}$$

$$= \lambda E[(X+1)^{n-1}]$$

Now, since $E[X] = \lambda$, this gives

$$E[X^2] = \lambda E[X+1]$$

$$= \lambda[E[X]+1] \qquad \text{Cor. 3 of 'Random Variables' notes}$$

$$= \lambda(\lambda+1)$$

$$= \lambda^2 + \lambda$$

Therefore,

$$E[X^3] = \lambda E[(X+1)^2]$$

$$= \lambda E[X^2 + 2X + 1]$$

$$= \lambda[E[X^2] + 2E[X] + 1]$$

$$= \lambda[(\lambda^2+\lambda) + 2\lambda + 1]$$

$$= \boxed{\lambda^3 + 3\lambda^2 + \lambda}$$

$$= \boxed{\lambda(\lambda^2 + 3\lambda + 1)} = \lambda\left(\lambda - \frac{-3+\sqrt{5}}{2}\right)\left(\lambda - \frac{-3-\sqrt{5}}{2}\right)$$

alternatively:

$$E[X+1] = E[X]+1$$

$$= \lambda + 1$$

$$\Longrightarrow E[X^3] = \lambda E[(X+1)^2]$$

exercise!

$$= \lambda[\lambda E[(X+2)] + E[X+1]]$$

$$= \lambda(\lambda(\lambda+2) + \lambda + 1)$$

$$= \lambda(\lambda^2 + 3\lambda + 1)$$

Chapter 5 Problems

(1) X has pdf $f(x) = \begin{cases} c(1-x^2), & -1<x<1 \\ 0, & \text{else} \end{cases}$

(a)
$$1 = \int_{-1}^{1} f(x)\,dx$$
$$= \int_{-1}^{1} c(1-x^2)\,dx$$
$$= c\left[x - \tfrac{1}{3}x^3\right]_{-1}^{1}$$
$$= c\left[(1-\tfrac{1}{3}) - (-1+\tfrac{1}{3})\right]$$
$$= c\left[2 - \tfrac{2}{3}\right]$$
$$= \frac{4c}{3}$$

$$\Longrightarrow \boxed{c = \frac{3}{4}}$$

(b) $\boxed{F(a) = 0}$ if $\boxed{a \le -1}$

$$\boxed{F(a) = \frac{3}{4}\int_{-1}^{a} 1 - x^2\,dx = \frac{3}{4}\left[x - \tfrac{1}{3}x^3\right]_{-1}^{a} = \frac{3}{4}\left(a - \tfrac{1}{3}a^3 + \tfrac{2}{3}\right),\ a \in (-1,1)}$$

$\boxed{F(a) = 1,\ a \ge 1}$

If $$f(x) = \begin{cases} c(2x - x^2), & 0 < x < 5/2 \\ 0, & \text{else,} \end{cases}$$

then

$$\int_0^{5/2} f(x)\,dx = c\left[x^2 - \tfrac{1}{3}x^3\right]_0^{5/2}$$

$$= c\left(\frac{25}{4} - \frac{125}{24}\right)$$

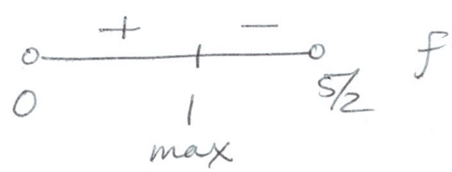
$$= \frac{25c}{24} = 1$$

$$\Longrightarrow \quad c = \frac{24}{25}$$

and since $$f'(x) = \frac{24}{25}(2 - 2x)$$

$$= \frac{48}{25}(1 - x)$$

$$= 0 \Longrightarrow x = 1$$



$$\Rightarrow f(0) = 0$$

$$f(1) = \frac{24}{25}$$

$$f\left(\tfrac{5}{2}\right) = \frac{24}{25}\cdot\frac{5}{2}\underbrace{\left(2 - \tfrac{5}{2}\right)}_{= -1/2} < 0 \quad \text{min}$$

$\boxed{\text{No!}}$

(3) $f(x) = \begin{cases} c(2x - x^3), & 0 < x < 5/2 \\ 0, & \text{else} \end{cases}$

$$\Rightarrow \int_0^{5/2} f(x)\,dx = c\left[x^2 - \tfrac{1}{4}x^4\right]_0^{5/2}$$

$$= c\left(\tfrac{25}{4} - \tfrac{625}{64}\right)$$

$$= c\left(\tfrac{-225}{64}\right)$$

$$= 1$$

$$\Rightarrow \quad c = \frac{-64}{225}$$

But then $f(x) = \begin{cases} \frac{64}{225}(x^3 - 2x), & 0 < x < 5/2 \\ 0, & \text{else} \end{cases}$

$$\Rightarrow \underline{f(1) = -\tfrac{64}{225} < 0}$$

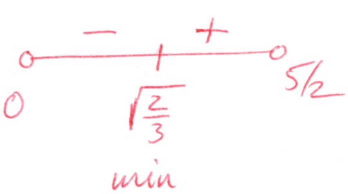
$\boxed{\text{so } f \text{ is } \underline{\text{not}} \ \underline{\text{nonnegative}}, \text{ so } f \ \underline{\text{cannot}} \text{ be a density!}}$

$*$ in fact, $g(x) = x^3 - 2x$

has $g'(x) = 3x^2 - 2$

$= 0$

$\Rightarrow x = \sqrt{\frac{2}{3}}$



$g\left(\sqrt{\tfrac{2}{3}}\right) = \left(\tfrac{2}{3}\right)^{3/2} - 2\left(\tfrac{2}{3}\right)^{1/2} < 0$

(10) (a)

$$P(A) = P\big((5 < X\ 15) \sqcup (20 < X < 30) \sqcup (35 < X\ 45) \sqcup (50 < X < 60)\big)$$

$$= \frac{1}{60}\big[(15-5) + (30-20) + (45-35) + (60-50)\big]$$

$$= \frac{4 \cdot 10}{60}$$

$$= \boxed{2/3}$$

minutes past 7am

since $f(x) = \begin{cases} \frac{1}{60}, & 0 \le x \le 60 \\ 0, & \text{else} \end{cases}$

(b)

$$P(A) = P\big((10 < X < 15) \sqcup (20 < X < 30) \sqcup (35 < X < 45) \sqcup (50 < X < 60) \sqcup (65 < X < 70)\big)$$

$$= \frac{1}{60}(5 + 3 \cdot 10 + 5)$$

$$= \frac{40}{60}$$

$$= \boxed{\frac{2}{3}}$$

(19) X normal with $\mu = 5$, $P(X > 9) = 0.2$

$$\Longrightarrow \quad 0.2 = P\left(\frac{X-5}{\sigma} > \frac{9-5}{\sigma}\right)$$

$$= P(Z > 4/\sigma)$$

$$\Longrightarrow \quad P(Z \le 4/\sigma) = 1 - 0.2 = 0.8$$

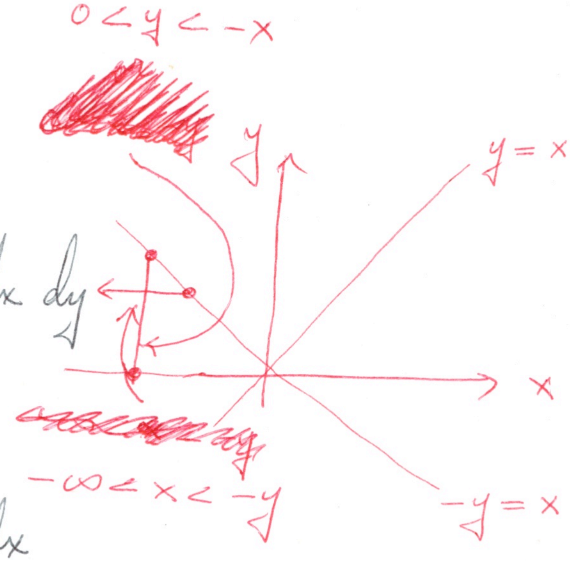
$$\Longrightarrow \quad \frac{4}{\sigma} \approx 0.84 \quad \text{(from table)}$$

$$\Longrightarrow \quad \sigma \approx \frac{4}{0.84} = 4.76190\overline{476190}$$

$$\Longrightarrow \quad \boxed{\sigma^2 = Var(X) = \left(\frac{4}{0.84}\right)^2 \approx 22.6757}$$

Ch. 5 Theoretical:

(2) $\displaystyle\int_0^\infty P(Y < -y)\, dy$

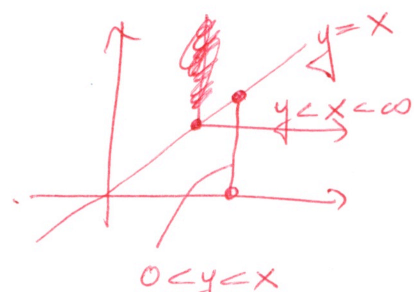


$$= \int_0^\infty \int_{-\infty}^{-y} f_Y(x)\, dx\, dy$$

$$= \int_{-\infty}^{0} \int_0^{-x} f_Y(x)\, dy\, dx$$

$$= \int_{-\infty}^{0} \left[s \right]_0^{-x} f_Y(x)\, dx$$

$$= -\int_{-\infty}^{0} x f_Y(x)\, dx$$



Similarly, $\displaystyle\int_0^\infty P(Y > y)\, dy = \int_0^\infty \int_y^\infty f_Y(x)\, dx\, dy$

$$= \int_0^\infty \int_0^x f_Y(x)\, dy\, dx$$

$$= \int_0^\infty x f_Y(x)\, dx$$

Therefore,

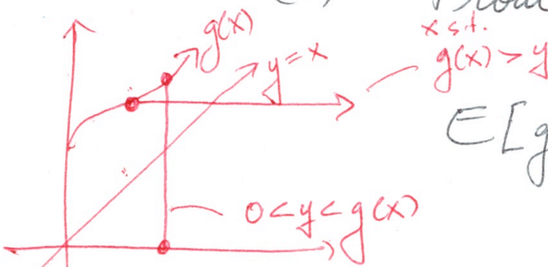
$$\int_0^\infty P(Y>y)\,dy - \int_0^\infty P(Y<-y)\,dy$$

$$= \int_0^\infty x f_Y(x)\,dx + \int_{-\infty}^0 x f_Y(x)\,dx$$

$$= \int_{-\infty}^\infty x f_Y(x)\,dx$$

$$= E[Y].$$

(3) From (2), for $g(x) \geq 0$



$$E[g(X)] = E[Y] = \int_0^\infty P(Y>y)\,dy - \int_0^\infty P(Y<-y)\,dy$$

$$= \int_0^\infty P(g(X)>y)\,dy - \int_0^\infty P(g(X)<-y)\,dy$$

$$= \int_0^\infty \int_{g^{-1}(y,\infty)} f(x)\,dx\,dy$$

$$= \int_{g^{-1}(0,\infty)} \int_0^{g(x)} f(x)\, dy\, dx$$

$$= \int_{g^{-1}(0,\infty)} g(x) f(x)\, dx$$

$$= \int_{-\infty}^{\infty} g(x) f(x)\, dx \qquad \left(\text{since } g(x) \geq 0 \iff g^{-1}[0,\infty) = \mathbb{R} \right)$$