HW 3

Chapter 3 Problems:

(5) Urn with 6 white, 9 Black balls.

Let E = 1st 2 white

F = last 2 black

$$\Longrightarrow \boxed{P(E \cap F) = P(F \mid E)P(E) = \frac{\binom{9}{2}}{\binom{13}{2}} \cdot \frac{\binom{6}{2}}{\binom{15}{2}}}$$

$$= \frac{9!}{\cancel{2!}\,7!} \cdot \frac{\cancel{2!}\,11!}{\cancel{13!}} \cdot \frac{\cancel{6!}}{\cancel{2!}\,4!} \cdot \frac{\cancel{2!}\,\cancel{13!}}{15!}$$

$$= \frac{\overset{3}{\cancel{9}} \cdot \overset{4}{\cancel{8}} \cdot \cancel{7} \cdot \overset{2}{\cancel{6}} \cdot \cancel{5} \cdot \cancel{4!}}{\cancel{7} \cdot \cancel{4!} \cdot \underset{3}{\cancel{15}} \cdot 13 \cdot \underset{6\,3}{\cancel{12}}}$$

$$= \boxed{\frac{6}{91} \approx 0.06593}$$

(8) Let E = 1st child (older) is a girl

F = 2nd child (younger) is a girl

$$\Longrightarrow \; P(F|E) = \frac{P(E\cap F)}{P(E)} = \frac{\frac{1}{2}\cdot\cancel{\frac{1}{2}}}{\cancel{\frac{1}{2}}} = \boxed{\frac{1}{2}}$$

or note simply that E & F are independent,

so $P(E\cap F) = P(E)P(F) \left(= \frac{1}{2}\cdot\frac{1}{2}\right)$

so that $P(F|E) = \dfrac{\cancel{P(E)}P(F)}{\cancel{P(E)}} = P(F)$.

$= 1/2$

(22) (a) $P(\underbrace{\text{no 2 dice land on the same number}})$

call this F

$$= \frac{6\cdot 5\cdot 4}{6\cdot 6\cdot 6} = \boxed{\frac{5}{9}}$$

(b) $P((B<Y<R)|F) = \boxed{\dfrac{1}{3!}}$ one way to get $B<Y<R$ out of $3!$ poss. permutations

(c) $P(B<Y<R) = P((B<Y<R)|F)\,P(F)$

$$= \frac{1}{6}\cdot\frac{5}{9} = \boxed{\frac{5}{54}}$$

* Could also do this directly: 4 choices for R (3, 4, 5, 6)

& $\binom{3-1}{2} + \binom{4-1}{2} + \binom{5-1}{2} + \binom{6-1}{2}$ slots for B & Y, etc.

(59) (a) $P(\overbrace{H,H,H,H}^{\text{1st 4}}) = P(H)^4 = \boxed{p^4}$

(b) $P(\overbrace{T,H,H,H}^{\text{1st 4}}) = \boxed{(1-p)p^3}$

(c) $P(\overbrace{THHH}^{\text{anywhere}} \text{ before } \overbrace{HHHH}^{\text{anywhere}})$

$= 1 - P(\overbrace{HHHH}^{\text{anywhere}} \text{ before 1st } T)$

$= 1 - P(\underbrace{HHHH}_{\text{1st 4}})$

$= \boxed{1 - p^4}$

Chapter 3 Theoretical:

(5)(a) $E \cap F = \emptyset \implies$

$$P(E \mid E \sqcup F) = \frac{P(\overbrace{E \cap (E \sqcup F)}^{= (E\cap E) \sqcup (E \cap F) = E \sqcup \emptyset = E})}{P(E \sqcup F)} = \frac{P(E)}{P(E)+P(F)}$$

(b) $\left(i \neq j \implies E_i \cap E_j = \emptyset\right) \implies$

$$P(E_j \mid \bigsqcup_{i=1}^{\infty} E_i) = \frac{P(\overbrace{E_j \cap (\bigsqcup_{i=1}^{\infty} E_i)}^{= \bigsqcup_{i=1}^{\infty} (E_j \cap E_i) = E_j})}{P(\bigsqcup_{i=1}^{\infty} E_i) = \sum_{i=1}^{\infty} P(E_i)}$$

Ch. 4 Problems:

(2) Roll 2 fair dice, $S = \{(i,j) \mid i,j = 1,\ldots,6\}$,

$X : S \to \mathbb{R}$, $X(i,j) = ij$.

Since $ij = ji$, we compute

$X(1,1) = 1$		
$X(1,2) = 2$	$X(2,2) = 4$	
$X(1,3) = 3$	$X(2,3) = 6$	$X(3,3) = 9$
$X(1,4) = 4$	$X(2,4) = 8$	$X(3,4) = 12$
$X(1,5) = 5$	$X(2,5) = 10$	$X(3,5) = 15$
$X(1,6) = 6$	$X(2,6) = 12$	$X(3,6) = 18$

$X(4,4) = 16$		
$X(4,5) = 20$	$X(5,5) = 25$	
$X(4,6) = 24$	$X(5,6) = 30$	$X(6,6) = 36$

Therefore,

$P(X=1) = P((1,1)) = \frac{1}{36}$

$P(X=2) = P((1,2),(2,1)) = \frac{2}{36}$

$P(X=3) = \frac{2}{36}$

$P(X=4) = \frac{3}{36}$

$P(X=5) = \frac{2}{36}$

$P(X=6) = \frac{4}{36}$

$P(X=7) = 0$

$P(X=8) = \frac{2}{36}$

$P(X=9) = \frac{1}{36}$

$P(X=10) = \frac{2}{36}$

$P(X=11) = 0$

$P(X=12) = \frac{4}{36}$

$P(X=13) = 0$

$P(X=14) = 0$

$P(X=15) = 2/36$

$P(X=16) = 1/36$

$P(X=17) = 0$

$P(X=18) = 2/36$

$P(X=19) = 0$

$P(X=20) = 2/36$

$P(X=21) = 0$

$P(X=22) = 0$

$P(X=23) = 0$

$P(X=24) = 2/36$

$P(X=25) = 1/36$

$P(X=30) = 2/36$

$P(X=36) = 1/36$

(14) $P(X=0) = P(\text{player \#1 loses to player \#2})$

$= P(i<j) \quad (1 \le i < j \le 5)$

$= \boxed{\frac{1}{2}}$ (there are $\binom{5}{2} = 10$ ways to choose $i<j$, but only 2 ways to distribute them to the players)

$P(X=1) = P(\text{player \#1 beats \#2 but loses to \#3})$

$= P(i_2 < i_1 < i_3)$

player #2's #

$=$ $\binom{5}{3}$ ways to choose $i_2 < i_1 < i_3$ out of 5, but only $3!$ to order them (& only 1 of these is correct)

$= \boxed{\frac{1}{3!} = \frac{1}{6}}$

$$P(X=2) = P(i_2 < i_3 < i_1 < i_4$$

or

$$i_3 < i_2 < i_1 < i_4)$$

$$= \boxed{\frac{2}{4!} = \frac{1}{12}}$$

$$P(X=3) = P(i_2 < i_3 < i_4 < i_1 < i_5$$

or

$$i_3 < i_2 < i_4 < i_1 < i_5$$

or

$\vdots$

$3! = 6$ ways to order i_2, i_3, i_4)

$$= \boxed{\frac{3!}{5!} = \frac{6}{5 \cdot \cancel{24}_{4}} = \frac{1}{20}}$$

$$P(X=4) = \boxed{1/5}$$

(17) $$F(b) = \begin{cases} 0 & \longleftrightarrow \quad b<0 \\ b/4 & \longleftrightarrow \quad 0 \le b < 1 \\ 1/2 + (b-1)/4 & \leftrightarrow \quad 1 \le b < 2 \\ 11/12 & \longleftrightarrow \quad 2 \le b < 3 \\ 1 & \longleftrightarrow \quad 3 \le b \end{cases}$$

(a) $$P(X=1) = P(X \le 1) - P(X \le 0)$$

$$= F(1) - F(0)$$

$$= \left(\frac{1}{2} + \frac{\cancel{b}-1}{4}\right) - \cancel{\frac{b}{4}}$$

$$= \boxed{\frac{1}{4}}$$

$$P(X=2) = P(X \le 2) - P(X \le 1)$$

$$= \frac{11}{12} - \left(\frac{1}{2} + \frac{b-1}{4}\right)$$

$$= \frac{8}{12} - \frac{b}{4}$$

$$= \boxed{\frac{2}{3} - \frac{b}{4}}$$

$$P(X=3) = P(X \le 3) - P(X \le 2)$$

$$= 1 - \frac{11}{12}$$

$$= \boxed{\frac{1}{12}}$$

(b) $P(\frac{1}{2} X < \frac{3}{2}) = P(X \le 1) - P(X \le 0)$

$$= P(X=1)$$

$$= \boxed{\frac{1}{4}}$$

(28) $E[X] = \sum_{i=0}^{3} i\,P(X=i)$ where $X = \#$ defective

$$= 1 \cdot P(X=1) + 2P(X=2) + 3P(X=3)$$

$$= \frac{1 \cdot \binom{4}{1}\binom{16}{2}}{\binom{20}{3}} + \frac{2 \cdot \binom{4}{2}\binom{16}{1}}{\binom{20}{3}} + \frac{3 \cdot \binom{4}{3}\binom{16}{0}}{\binom{20}{3}}$$

$$= \frac{3!\,17!}{20!}\left(4 \cdot 8 \cdot 15 + 2 \cdot 6 \cdot 16 + 3 \cdot 4\right)$$

$$= \frac{6}{20 \cdot 19 \cdot 18}\left(4 \cdot 3(8 \cdot 5 + 2 \cdot 2 \cdot 4 + 1)\right)$$

$$= \frac{\cancel{6} \cdot \cancel{4} \cdot \cancel{3} \cdot \overset{3}{\cancel{57}}}{{}_{5}\cancel{20} \cdot \cancel{19} \cdot \cancel{18}} = \boxed{\frac{3}{5}}$$

(40) $P(X \geq 4) = P(X=4) + P(X=5)$

$$= \binom{5}{4}\left(\frac{1}{3}\right)^4\left(\frac{2}{3}\right) + \left(\frac{1}{3}\right)^5$$

$$= \boxed{\frac{11}{243}}$$