HW 1

Chapter 1 Problems

(5) $8 \cdot 2 \cdot 9 = \boxed{144}$

and

$2 \cdot 9 = \boxed{18}$

(15) $\underbrace{\binom{10}{5}}_{\text{\# ways to choose 5 women}} \cdot \underbrace{\binom{12}{5}}_{\text{\# ways to choose 5 men}} \cdot \underbrace{5!}_{\text{\# ways to pair}} = \boxed{23,950,080}$

\# ways to pair (for man #1, there are 5 women, for man #2, there are 4, etc.)

(17) $\underbrace{\binom{10}{7}}_{\text{\# ways to choose 7 of 10 kids}} \cdot \underbrace{7!}_{\text{\# ways to distribute 7 gifts}} = 604,800$

Alternative soln:

$\underbrace{10 \cdot 9 \cdot 8 \cdots 4} = 604,800$

10 choices 1st kid to get gift

9 choices 2nd kid to get gift

$\vdots$

note $10 \cdot 9 \cdots 4 = \frac{10!}{3!} = \binom{10}{7} \cdot 7!$

(20) (a) 8 friends, 5 invited, 2 fending:

$$\underbrace{\binom{8}{5}}_{\substack{\text{\# ways to}\\ \text{choose any}\\ \text{5 of 8}}} - \underbrace{\binom{2}{2}}_{\substack{\text{\# ways}\\ \text{to choose}\\ \text{the 2}\\ \text{fending}}}\underbrace{\binom{6}{3}}_{\substack{\text{\# ways}\\ \text{to choose}\\ \text{3 of remaining}\\ \text{6 non-fending}}} = 56 - 20 = \boxed{36}$$

Alternatively,

$$\underbrace{\binom{6}{5}}_{\substack{\text{\# ways to}\\ \text{choose 5 of}\\ \text{6 non-fending}}} + \underbrace{\binom{2}{1}}_{\substack{\text{\# ways}\\ \text{to choose}\\ \text{one of}\\ \text{2 fending}}}\underbrace{\binom{6}{4}}_{\substack{\text{\# ways to}\\ \text{choose 4}\\ \text{of remaining}\\ \text{6 non-}\\ \text{fending}}} = 6 + 30 = \boxed{36}$$

(b) 2 friends will only attend together:

$$\underbrace{\binom{8}{5}}_{\text{rest}} - \underbrace{\binom{2}{1}\binom{6}{4}}_{\text{besties}} = 56 - 30 = \boxed{26}$$

OR

$$\underbrace{\binom{2}{2}\binom{6}{3}}_{\text{besties}} + \underbrace{\binom{6}{5}}_{\text{rest}} = 20 + 6 = \boxed{26}$$

(24) $(3x^2+y)^5 = \sum_{k=0}^{5} \binom{5}{k} (3x^2)^k y^{5-k}$

$$= \overbrace{\binom{5}{0}}^{=1} y^5 + \overbrace{\binom{5}{1}}^{=5} (3x^2) y^4 + \overbrace{\binom{5}{2}}^{=10} (3x^2)^2 y^3$$

$$+ \underbrace{\binom{5}{3}}_{=10} (3x^2)^3 y^2 + \underbrace{\binom{5}{4}}_{=5} (3x^2)^4 y + \underbrace{\binom{5}{5}}_{=1} (3x^2)^5$$

$$= y^5 + 5(3x^2)y^4 + 10(3x^2)^2 y^3 + 10(3x^2)^3 y^2 + 5(3x^2)^4 y + (3x^2)^5$$

$$= \boxed{y^5 + 15x^2y^4 + 90x^4y^3 + 270x^6y^2 + 405x^8y + 243x^{10}}$$

(26) $(x_1+2x_2+3x_3)^4 = \sum_{n_1+n_2+n_3=4} \binom{4}{n_1,n_2,n_3} x_1^{n_1} x_2^{n_2} x_3^{n_3}$

* There are $\binom{4+3-1}{3-1} = \binom{6}{2} = 15$ solutions to $n_1+n_2+n_3=4$ (by Prop. 6.2, p. 13), given by

$$\underbrace{\binom{4}{4,0,0} = \binom{4}{0,4,0} = \binom{4}{0,0,4}}_{\text{3 ways to have one 4th power}} = \frac{4!}{4!\,0!\,0!} = 1$$

$$\underbrace{\binom{4}{3,1,0} = \binom{4}{3,0,1} = \binom{4}{1,3,0} = \binom{4}{0,3,1} = \binom{4}{1,0,3} = \binom{4}{0,1,3}}_{\text{6 ways to have one 3rd power \& one 1st power}} = \frac{4!}{3!\,1!\,0!} = 4$$

$$\underbrace{\binom{4}{2,2,0} = \binom{4}{2,0,2} = \binom{4}{0,2,2}}_{\text{3 ways to have two squares}} = \frac{4!}{2!\,2!\,0!} = 6$$

$$\underbrace{\binom{4}{2,1,1} = \binom{4}{1,2,1} = \binom{4}{1,1,2}}_{\text{3 ways to have one square \& two 1st powers}} = \frac{4!}{1!\,1!\,2!} = 12$$

Plugging these numbers into the summation gives:

$$(x_1+2x_2+3x_3)^4 = \sum_{n_1+n_2+n_3=4} \binom{4}{n_1,n_2,n_3} x_1^{n_1} x_2^{n_2} x_3^{n_3}$$

$$= \binom{4}{4,0,0} x_1^4 + \binom{4}{0,4,0} (2x_2)^4 + \binom{4}{0,0,4} (3x_3)^4$$

$$+ \binom{4}{3,1,0} x_1^3 (2x_2) + \binom{4}{3,0,1} x_1^3 (3x_3)$$

$$+ \binom{4}{1,3,0} x_1 (2x_2)^3 + \binom{4}{0,3,1} (2x_2)^3 (3x_3)$$

$$+ \binom{4}{1,0,3} x_1 (3x_3)^3 + \binom{4}{0,1,3} (2x_2)(3x_3)^3$$

$$+ \binom{4}{2,2,0} x_1^2 (2x_2)^2 + \binom{4}{2,0,2} x_1^2 (3x_3)^2$$

$$+ \binom{4}{0,2,2} (2x_2)^2 (3x_3)^2 + \binom{4}{2,1,1} x_1^2 (2x_2)(3x_3)$$

$$+ \binom{4}{1,2,1} x_1 (2x_2)^2 (3x_3) + \binom{4}{1,1,2} x_1 (2x_2)(3x_3)^2$$

$$= \boxed{\begin{aligned} & x_1^4 + 16x_2^4 + 81x_3^4 + 8x_1^3x_2 + 12x_1^3x_3 \\ & + 32x_1x_2^3 + 96x_2^3x_3 + 108x_1x_3^3 + 216x_2x_3^3 \\ & + 24x_1^2x_2^2 + 54x_1^2x_3^2 + 216x_2^2x_3^2 \\ & + 72x_1^2x_2x_3 + 144x_1x_2^2x_3 + 216x_1x_2x_3^2 \end{aligned}}$$

Chapter 1 Theoretical Exercises:

(5) Fix $0 \le k \le n$, then the number of vectors $(x_1, \dots, x_n)$ satisfying

(i) $x_i = 0$ or 1 for all $i = 1, \dots, n$

(ii) $\sum_{i=1}^{n} x_i = k$

is just

$$\binom{n}{k} = \#\text{ of ways to place } k \text{ 1's in } n \text{ slots}$$

Therefore,

$$\boxed{\sum_{i=1}^{n} x_i \ge k}$$

$\iff$

$$\sum_{i=1}^{n} x_i = k$$

or

$$\sum_{i=1}^{n} x_i = k+1$$

$$\vdots$$

$$\sum_{i=1}^{n} x_i = n$$

$\iff$ $\boxed{\binom{n}{k} + \binom{n}{k+1} + \cdots + \binom{n}{n} = \sum_{j=k}^{n} \binom{n}{j} \text{ Solutions}}$

(6) The number of vectors $(x_1, \dots, x_k)$ satisfying

(i) $x_i \in \mathbb{N}$

(ii) $1 \leq x_i \leq n$

(iii) $x_1 < x_2 < \cdots < x_k$

is $\boxed{\binom{n}{k}}$ because we must choose k of n slots in which to place each x_i:

choose k slots for the x_i

$$\underset{1}{\underline{\quad}}\ \ \underset{2}{\underline{\quad}}\ \ \cdots\ \ \underset{n-1}{\underline{\quad}}\ \ \underset{n}{\underline{\quad}}$$

slot #s

(12)(a) $\displaystyle\sum_{k=1}^{n} k\binom{n}{k} = n \cdot 2^{n-1}$

<u>Combinatorial pf:</u> for fixed k, there are $\binom{n}{k}$ ways to choose a committee of size k out of n candidates, & this is followed by k ways to

select a committee chair from those k, so there are

$$k\binom{n}{k}$$

ways to pick a k-committee-out-of-n + a chair.

Summing over $k = 1, 2, \ldots, n$ gives us all poss. k-committee + chair possibilities.

On the other hand, we could first determine the committee size, k, and there are n possibilities, followed by a choice of membership of the remaining $n-1$ ~~members~~ people (1 means they're *in*, 0 they're *out* of the committee) after we've picked its chair:

# of choices of committee size k + chair $\rightarrow$ $n \cdot 2^{n-1}$ $\leftarrow$ whether each of the $n-1$ left are in or out

(13) For $n > 0$,

$$\sum_{i=0}^{n} (-1)^i \binom{n}{i} = 0$$

pf: $\displaystyle\sum_{i=0}^{n} (-1)^i \binom{n}{i} = \sum_{i=0}^{n} \binom{n}{i} (-1)^i \cdot 1^{n-i}$

$= (1-1)^n$

$= 0^n$

$= 0$ $\boxtimes$