Hw 9 Evens

Chapter 10

(2) $\boxed{\sum_{i=1}^{n} i^2 = \frac{n(n+1)(2n+1)}{6}}$

pf: For $n=1$ this follows directly:

$$\sum_{i=1}^{n} i^2 = \sum_{i=1}^{1} i^2 = 1^2 = 1 = \frac{6}{6} = \frac{1\cdot 2\cdot 3}{6}$$

$$= \frac{n(n+1)(2n+1)}{6}$$

Now suppose $\sum_{i=1}^{n} i^2 = \frac{n(n+1)(2n+1)}{6}$ for some $n \geq 1$ & consider $\sum_{i=1}^{n+1} i^2$:

$$\sum_{i=1}^{n+1} i^2 = \sum_{i=1}^{n} i^2 + (n+1)^2$$

$$= \frac{n(n+1)(2n+1)}{6} + \frac{6(n+1)^2}{6}$$

$$= \frac{(n+1)[n(2n+1) + 6(n+1)]}{6} \quad \text{factor } (n+1)$$

$$= \frac{(n+1)[2n^2 + 7n + 6]}{6} \quad \text{FOIL}$$

$$= \frac{(n+1)(n+2)(2n+3)}{6} \quad \text{factor}$$

$$= \frac{(n+1)((n+1)+ \cdot (2(n+1)+}{6}$$

(4) $\displaystyle\sum_{i=1}^{n} i(i+1) = \frac{n(n+1)(n+2)}{3}$

by induction

pf: For $n=1$, $\displaystyle\sum_{i=1}^{n} i(i+1) = 1(1+1) = 2$

$$= \frac{6}{3}$$
$$= \frac{1\cdot 2\cdot 3}{3}$$
$$= \frac{n(n+1)(n+2)}{3}$$

Now suppose $\displaystyle\sum_{i=1}^{n} i(i+1) = \frac{n(n+1)(n+2)}{3}$

for some $n \geq 1$, & observe

$$\sum_{i=1}^{n+1} i(i+1) = \sum_{i=1}^{n} i(i+1) + (n+1)(n+1+1)$$

$$= \frac{n(n+1)(n+2)}{3} + \frac{3(n+1)(n+2)}{3}$$

$$= (n+3)(n+1)(n+2)$$

$$= \frac{(n+1)((n+1)+1)((n+1)+2)}{3}$$

using other formulas:

pf#2: $\displaystyle\sum_{i=1}^{n} i(i+1) = \sum_{i=1}^{n} i^2 + \sum_{i=1}^{n} i$

$$= \frac{n(n+1)(2n+1)}{6} + \frac{n(n+1)}{2}\cdot\frac{3}{3}$$

$$= \frac{2n^3+3n^2+n+3n^2+3n}{6}$$

$$= \frac{2n^3+6n^2+4n}{6}$$

$$= \frac{\cancel{2}n(n^2+3n+2)}{\cancel{6}\,3}$$

$$= \frac{n(n+1)(n+2)}{3} \checkmark$$

(8) $$\boxed{\sum_{i=1}^{n} \frac{i}{(i+1)!} = 1 - \frac{1}{(n+1)!}}$$

by induction:

pf: For $n=1$,

base step

$$\sum_{i=1}^{n} \frac{i}{(i+1)!} = \sum_{i=1}^{1} \frac{i}{(i+1)!} = \frac{1}{(1+1)!} = \frac{1}{2}$$

$$= 1 - \frac{1}{2} = 1 - \frac{1}{(1+1)!}$$

$$= 1 - \frac{1}{(n+1)!}$$

Now suppose true for some $n \geq 1$, & consider

inductive step

$$\sum_{i=1}^{n+1} \frac{i}{(i+1)!} = \sum_{i=1}^{n} \frac{i}{(i+1)!} + \frac{n+1}{(n+1+1)!}$$

$$= 1 - \frac{1}{(n+1)!} + \frac{n+1}{(n+2)!}$$

$\cdot \frac{n+2}{n+2}$

$$= 1 + \frac{-\cancel{n}-2+\cancel{n}+1}{(n+2)!}$$

$$= 1 - \frac{1}{((n+1)+1)!}$$

direct proof: pf #2:

$$\left(\sum_{i=1}^{n} \frac{i}{(i+1)!}\right) + \frac{1}{(n+1)!}$$

$$= \left(\frac{1}{2!} + \frac{2}{3!} + \cdots + \frac{n}{(n+1)!}\right) + \frac{1}{(n+1)!}$$

$$\underbrace{\frac{n}{(n+1)!} + \frac{1}{(n+1)!}}_{= \frac{n+1}{(n+1)!} = \frac{1}{n!}}$$

$$\frac{n-1}{(n-1+1)!} = \frac{n-1}{n!}$$

$$\frac{n-1}{n!} + \frac{1}{n!} = \frac{n}{n!} = \frac{1}{(n-1)!}$$

$$= \frac{1}{2!} + \frac{2}{3!} + \cdots + \underbrace{\frac{(n-1)}{n!} + \frac{n+1}{(n+1)!}}_{= \frac{n-1+1}{n!} = \frac{n}{n!} = \frac{1}{(n-1)!}}$$

$$= \frac{1}{2!} + \frac{2}{3!} + \cdots + \frac{n-3}{(n-2)!} + \underbrace{\frac{n-2}{(n-1)!} + \frac{1}{(n-1)!}}_{= \frac{n-1}{(n-1)!} = \frac{1}{(n-2)!}}$$

$$= \frac{n-3+1}{(n-2)!} = \frac{1}{(n-3)!}$$

$\vdots$

$$= \frac{1}{2!} + \frac{1}{2!} = 1$$

(20) $$\boxed{\left(\sum_{i=1}^{n} i\right)^2 = \sum_{i=1}^{n} i^3}$$

pf: This was proved recursively in class & on exams, & can be rephrased

$$\boxed{\sum_{i=1}^{n} i = \frac{n\cdot(n+1)}{2}}$$

$$\Longrightarrow \boxed{\sum_{i=1}^{n} i^3 = \left(\frac{n(n+1)}{2}\right)^2}$$

but let's use induction here:

(1) For $n=1$, $\left(\sum_{i=1}^{n} i\right)^2 = 1^2 = \left(\frac{2}{2}\right)^2 = \left(\frac{1\cdot 2}{2}\right)^2$

$$= \left(\frac{n(n+1)}{2}\right)^2$$

$$= 1^3 = \sum_{i=1}^{n} i^3$$

(2) Now suppose true for $n \geq 1$, and consider:

$$\sum_{i=1}^{n+1} i^3 = \sum_{i=1}^{n} i^3 + (n+1)^3$$

$$= \left(\frac{n(n+1)}{2}\right)^2 + \frac{4(n+1)^3}{4}$$

$$= \frac{(n+1)^2\left[\underbrace{n^2 + 4(n+1)}_{= n^2+4n+4}\right]}{4} \quad = (n+2)^2$$

$$= \frac{(n+1)^2(n+2)^2}{4}$$

$$= \left(\frac{(n+1)((n+1)+1)}{2}\right)^2 \qquad (1)$$

and similarly, for $n=1$

$$\sum_{i=1}^{n} i = 1 = \frac{2}{2} = \frac{1\cdot 2}{2} = \frac{n(n+1)}{2}$$

& if $\sum_{i=1}^{n} i = \frac{n(n+1)}{2}$, then

$$\sum_{i=1}^{n+1} i = \sum_{i=1}^{n} i + (n+1)$$

factor

$$= \frac{n(n+1)}{2} + \frac{2(n+1)}{2}$$

$$= \frac{(n+1)(n+2)}{2}$$

$$= \frac{(n+1)((n+1)+1)}{2} \qquad (2)$$

Then square both sides of (2) & compare with (1).

(22) $\boxed{\prod_{i=1}^{n}\left(1-\frac{1}{2^i}\right) \geq \frac{1}{4} + \frac{1}{2^{n+1}}}$ for all $n \in \mathbb{N}$.

pf: For $n=1$, $\prod_{i=1}^{n}\left(1-\frac{1}{2^i}\right) = 1-\frac{1}{2} = \frac{1}{2}$

$$\geq \frac{1}{2}$$

$$= \frac{1}{4}+\frac{1}{4}$$

$$= \frac{1}{4}+\frac{1}{2^{n+1}}$$

Suppose true for $n \geq 1$ & consider

$$\prod_{i=1}^{n+1}\left(1-\frac{1}{2^i}\right) = \left(\prod_{i=1}^{n}\left(1-\frac{1}{2^i}\right)\right)\cdot\left(1-\frac{1}{2^{n+1}}\right)$$

$$\geq \left(\frac{1}{4}+\frac{1}{2^{n+1}}\right)\left(1-\frac{1}{2^{n+1}}\right)$$

$$= \frac{1}{4} + \underbrace{\frac{1}{2^{n+1}} - \frac{1}{4\cdot 2^{n+1}} - \frac{1}{2^{2n+2}}}_{=\frac{1}{2^{n+1}}\left(1-\frac{1}{4}-\frac{1}{2^{n+1}}\right)}$$

$$\frac{1}{4}+\frac{1}{2^{n+1}} \leq \frac{2}{4} = \frac{1}{2}$$

$$1-\frac{1}{4}-\frac{1}{2^{n+1}} \geq \frac{1}{2} \Leftarrow$$

$$\geq \frac{1}{4}+\frac{1}{2^{n+1}}\cdot\frac{1}{2}$$

$$= \frac{1}{4}+\frac{1}{2^{n+1+1}} \checkmark$$

(26) $\sum_{k=1}^{n} F_k^2 = F_n F_{n+1}$ where $F_1 = 1$

$F_2 = 1$

$F_3 = 2$

$\vdots$

$F_n = F_{n-1} + F_{n-2}$

pf: For $n=1$, $\sum_{k=1}^{n} F_k^2 = 1^2 = 1 \cdot 1 = F_n \cdot F_{n+1}$.

Suppose true for $n \geq 1$ & consider

$$\sum_{k=1}^{n+1} F_k^2 = \sum_{k=1}^{n} F_k^2 + F_{n+1}^2$$

$$= F_n F_{n+1} + F_{n+1}^2$$

$$= F_{n+1}(F_n + F_{n+1})$$

$$= F_{n+1} F_{\underbrace{n+2}_{=n+1+1}}$$

✓

Sec. 11.4

(2) $A = \{a, b, c\}$

$\Longrightarrow$ 1. $\{a, b, c\} = A$

2. $\{a\} \cup \{b, c\} = A$

3. $\{b\} \cup \{a, c\} = A$

4. $\{c\} \cup \{a, b\} = A$

5. $\{a\} \cup \{b\} \cup \{c\}. = A$

So $\boxed{\{\{a,b,c\}\}, \quad \{\{a\}; \{b,c\}\} \quad \{\{b\}, \{a,c\}\}, \quad \{\{c\}, \{a,b\}\} \quad \{\{a\}, \{b\}, \{c\}\}}$

5 partitions

(6) $\boxed{n \sim m \iff n = -m}$

(4) $\mathbb{Z}_6 = \{[0], [1], [2], [3], [4], [5]\}$

I'm going to use $\bar{0}$ for $[0]$, it's easier to write:

$+$	0	1	2	3	4	5
0	$\bar{0}$	$\bar{1}$	$\bar{2}$	$\bar{3}$	$\bar{4}$	$\bar{5}$
1	$\bar{1}$	$\bar{2}$	$\bar{3}$	$\bar{4}$	$\bar{5}$	$\bar{0}$
2	$\bar{2}$	$\bar{3}$	$\bar{4}$	$\bar{5}$	$\bar{0}$	$\bar{1}$
3	$\bar{3}$	$\bar{4}$	$\bar{5}$	$\bar{0}$	$\bar{1}$	$\bar{2}$
4	$\bar{4}$	$\bar{5}$	$\bar{0}$	$\bar{1}$	$\bar{2}$	$\bar{3}$
5	$\bar{5}$	$\bar{0}$	$\bar{1}$	$\bar{2}$	$\bar{3}$	$\bar{4}$

e.g. $4+5 = 9 \equiv 3 \mod 6$

$\Rightarrow [4]+[5] = [3]$.

$\cdot$	0	1	2	3	4	5
0	0	0	0	0	0	0
1	0	1	2	3	4	5
2	0	2	4	0	2	4
3	0	3	0	3	0	3
4	0	4	2	0	4	2
5	0	5	4	3	2	1

e.g. $3 \cdot 5 = 15 \equiv 3 \mod 6$

$\Leftrightarrow [3]\cdot[5] = [15] = [3]$

(6) $[a], [b] \in \mathbb{Z}_6$ & $[a][b] = [0]$.

It is not nec. that $[a] = [0]$, or $[b] = [0]$. e.g.

$$[2][3] = [6] = [0]$$

but $[2], [3] \neq [0]$.

$$
\begin{aligned}
[a][b] = [0] \text{ in } \mathbb{Z}_7 &\Rightarrow ab \equiv 0 \bmod 7 \\
&\Rightarrow 7 \mid ab \\
&\Rightarrow 7 \mid a \text{ or } 7 \mid b \quad \text{(bec. 7 is prime)} \\
&\Rightarrow a \equiv 0 \bmod 7 \text{ or } \\
&\qquad b \equiv 0 \bmod 7 \\
&\Rightarrow [a] = [0] \text{ or } [b] = [0].
\end{aligned}
$$

(8) $a, a', b, b' \in \mathbb{Z}$ w/ $a \equiv a' \bmod n$, $b \equiv b' \bmod n$

$\Rightarrow$ $a - a' = kn$, $b - b' = ln$ $\Rightarrow$

$$
\left.
\begin{aligned}
(a+b) - (a'+b') &= (a-a') + (b-b') \\
&= kn + ln \\
&= (k+l)n
\end{aligned}
\right\} \Rightarrow a+b \equiv a'+b' \bmod n
$$