HW 8 Evens

Chapter 8

(6) Let A, B, C be sets, and let us show that if $A \subseteq B$ then $A - C \subseteq B - C$.

pf: Let $a \in A - C$, so $a \in A$ but $a \notin C$. Since $a \in A$ & $A \subseteq B$, we know $a \in B$, but since $a \notin C$, $a \in B \subseteq C$.

(8) $A \cup (B \cap C) = (A \cup B) \cap (A \cup C)$

pf: Let $a \in A \cup (B \cap C)$, so that

$a \in A$ or $a \in B \cap C \Rightarrow a \in B$ & $a \in C$

If $a \in A$, then $a \in A \cup B$ and $a \in A \cup C$, so $a \in (A \cup B) \cap (A \cup C)$. If $a \in B \cap C$, then $a \in B$ & $a \in C$, so also $a \in A \cup B$ & $a \in A \cup C$, whence $a \in (A \cup B) \cap (A \cup C)$. Thus,

$$A \cup (B \cap C) = (A \cup B) \cap (A \cup C)$$

To show the reverse inclusion, let $a \in (A \cup B) \cap (A \cup C)$, so that

$$a \in A \cup B \quad \text{and} \quad a \in A \cup C$$

Since $a \in A \cup B$, we have

$$a \in A \quad \text{or} \quad a \in B \qquad (*)$$

<u>and</u> since $a \in A \cup C$,

$$a \in A \text{ or } a \in C \qquad (**)$$

If $a \in A$, then $a \in A \cup (B \cap C)$, while if $a \notin A$, then $a \in B$ and $a \in C$, by $(*)$ & $(**)$, so $a \in B \cap C$ & therefore $a \in A \cup (B \cap C)$.

Thus

$$(A \cup B) \cap (A \cup C) \subseteq A \cup (B \cap C)$$

& so

$$(A \cup B) \cap (A \cup C) = A \cup (B \cap C).$$

(10) Let $A, B \subseteq U$, & define complements

$$\bar{A} = U - A$$
$$\bar{B} = U - B$$
$$\overline{A \cap B} = U - (A \cap B)$$

Then,
$$\overline{A \cap B} = \bar{A} \cup \bar{B}$$

pf: Let $a \in \overline{A \cap B} = U - (A \cap B)$, so

$a \in U$ but $a \notin A \cap B$

There are three cases,

(i) $a \in A \subseteq U \Rightarrow a \notin B \Rightarrow a \in U - B$

(ii) $a \in B \subseteq U \Rightarrow a \notin A \Rightarrow a \in U - A$

(iii) $a \in U - (A \cup B) \Rightarrow a \in U - A$ and $a \in U - B$

In all 3 cases,

$$a \in (U - A) \cup (U - B) = \bar{A} \cup \bar{B}$$

Thus,
$$\overline{A \cap B} \subseteq \bar{A} \cup \bar{B}$$

For the reverse inclusion, let $a \in \overline{A} \cup \overline{B}$
$= (U-A) \cup (U-B)$.

Then $a \in U-A$ or $a \in U-B$, so $a \notin A$ or $a \notin B$, & consequently $a \notin A \cap B$. Thus, $a \in U$, but $a \notin A \cap B$, i.e.

$$\overline{A} \cup \overline{B} \subseteq \overline{A \cap B}.$$

(12) $A - (B \cap C) = (A-B) \cup (A-C)$

This is like #10, except that B, C need not be subsets of A. ~~But in that case let $U = A \cap (B \cup C) = (A \cap B) \cup (A \cap C)$ (cf. #9) & note $A - (B \cap C) = U - (B \cap C)$~~

Let $a \in A - (B \cap C)$, so $a \in A$ but $a \notin B \cap C$. There are 3 cases,

(i) $a \in B \Rightarrow a \notin C \Rightarrow a \in A - C$

(ii) $a \in C \Rightarrow a \notin B \Rightarrow a \in A - B$

(iii) $a \in A - (B \cup C) \Rightarrow a \in A - B$ & $a \in A - C$

So in all 3 cases,

$$a \in (A-B) \cup (A-C)$$

so

$$A - (B \cap C) \subseteq (A-B) \cup (A-C)$$

For the reverse inclusion, let $a \in (A-B) \cup (A-C)$

so

$$a \in A-B \text{ or } a \in A-C)$$

$$\Rightarrow a \notin B \text{ or } a \notin C \text{ but } a \in A$$

$$\rightarrow a \in A \text{ but } a \notin B \cap C$$

$$\Rightarrow a \in A - (B \cap C).$$

&

$$(A-B) \cup (A-C) \subseteq A - (B \cap C).$$

(14) $(A \cup B) - C = (A - C) \cup (B - C)$

pf. $\subseteq$: If $a \in (A \cup B) - C$, then $a \in A \cup B$ but $a \notin C$, so $a \in A$ or $a \in B$ but $a \notin C$. If $a \in A$, then $a \in A - C$, while if $a \in B$, then $a \in B - C$, & a is in one of these (bec. $a \in A$ or $a \in B$), so

$$a \in (A - C) \cup (B - C)$$

For $\supseteq$: Let $a \in (A - C) \cup (B - C)$, so

$$a \in A - C \text{ or } a \in B - C$$

$\Longrightarrow$ ~~[illegible]~~

$a \in A$ or $a \in B$, but $a \notin C$

$\Longrightarrow$ $a \in (A \cup B) - C$.

(16) $A \times (B \cup C) = (A \times B) \cup (A \times C)$

pf: $\subseteq$: let $(x,y) \in A \times (B \cup C)$

$= \{(x,y) \mid x \in A,\ y \in B \cup C\}$

Then

$x \in A,\ y \in B \cup C$

$\Longrightarrow x \in A,\ y \in B$ or $x \in A,\ y \in C$

$\Longrightarrow (x,y) \in A \times B$ or $(x,y) \in A \times C$

$\Longrightarrow (x,y) \in (A \times B) \cup (A \times C)$

For $\supseteq$: let $(x,y) \in (A \times B) \cup (A \times C)$

$\Longrightarrow (x,y) \in A \times B$ or $(x,y) \in A \times C$

$\Longrightarrow x \in A$ & $y \in B$ or $x \in A,\ y \in C$

$\Longrightarrow x \in A$ & $y \in B$ or C

$\Longrightarrow (x,y) \in A \times (B \cup C)$.

oops, this is (21)

~~(2)~~ $A \subseteq B \iff A - B = \emptyset$

<u>pf:</u> (1) $\Longrightarrow$: If $A-B \neq \emptyset$, then $\exists\, a \in A-B$, i.e. $\exists\, a \in A$ with $a \notin B$, so we <u>cannot</u> have $A \subseteq B$, because if we did we would be forced to admit $a \in B$, which contradicts $a \notin B$. Thus

$$(A \subseteq B) \Longrightarrow (A - B = \emptyset)$$

(matter of fact $A \subseteq B =$ "$a \in A \Rightarrow a \in B$"

while $\sim(A \subseteq B) = \sim($"$a \in A \Rightarrow a \in B$"$)$

$=$ "$a \in A$ & $a \notin B$"

$=$ "$a \in A - B \neq \emptyset$"

$=$

so $\sim(A-B=\emptyset) \iff \sim(A \subseteq B)$)

(2) $\Longleftarrow$: Suppose $\sim(A \subseteq B)$, i.e. $\exists\, a \in A-B$, then $A-B \neq \emptyset$, i.e. $A - B \cancel{\emptyset} = \emptyset \Rightarrow A \subseteq B$.

(22) $A \subseteq B \iff A \cap B = A$

pf: (1) Suppose $A \subseteq B$, & consider $a \in A$. Since $a \in A$ & $A \subseteq B$, $a \in B$, so $a \in A \cap B$, which shows

$$A \subseteq A \cap B$$

But clearly $A \cap B \subseteq A$, since any $a \in A \cap B$ is both in A & in B. Thus,

$$A = A \cap B$$

(2) Now suppose that $A \cap B = A$ & choose any $a \in A$. Since $A = A \cap B$, $a \in A \cap B$ too, so $a \in B$ also. Thus,

$$A \subseteq B$$

Chapter 9

(6) $(A\times B)\cap(C\times D) = (A\cap C)\times(B\cap D)$ True

pf: (1) $\subseteq$: Let $(x,y)\in(A\times B)\cap(C\times D)$,

so that $(x,y)\in A\times B$ & $(x,y)\in C\times D$. Then,

$x\in A,\ y\in B$ and $x\in C,\ y\in D$

$\Rightarrow$ $x\in A\cap C$ and $y\in B\cap D$

$\Rightarrow$ $(x,y)\in(A\cap C)\times(B\cap D)$

(2) $\supseteq$: Let $(x,y)\in(A\cap C)\times(B\cap D)$, so

$x\in A\cap C$ & $y\in B\cap D$

$\Rightarrow$ $x\in A,\ y\in B$ & $x\in C,\ y\in D$

$\Rightarrow$ $(x,y)\in(A\times B)\cap(C\times D)$.

(8) $A-(B\cup C) = (A-B)\cup(A-C)$ False

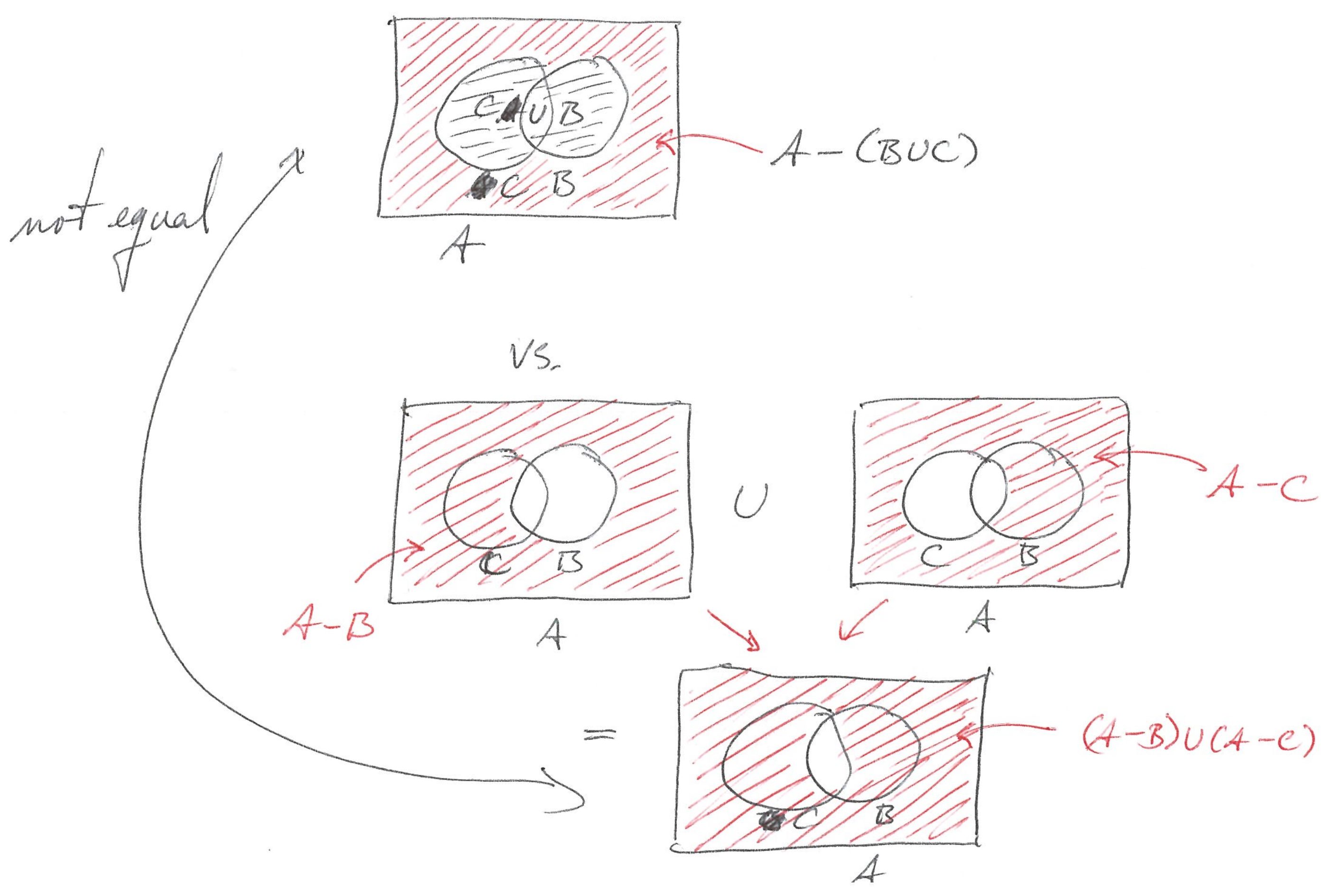
pf: This is false. Recall ch. 8 problems (12)-(13), which you proved:

$$(A-B)\cup(A-C) = A-(B\underset{\text{not }\cup}{\cap}C)$$

&

$$A-(B\cup C) = (A-B)\underset{\text{not }\cup}{\cap}(A-C)$$

ex.



<u>ex.</u> $A = \{a, b, c\}$, $B = \{a, b\}$, $C = \{b, c\}$

$$A - (B \cup C) = A - A = \emptyset$$

vs.

$$\begin{aligned}(A - B) \cup (A - C) &= \{c\} \cup \{a\} \\ &= \{a, c\} \\ &\neq \emptyset\end{aligned}$$

(10) $A \cap B = \emptyset \implies \mathcal{P}(A) - \mathcal{P}(B) \subseteq \mathcal{P}(A - B)$

is True

<u>pf:</u> $A \cap B = \emptyset \implies \mathcal{P}(A) \cap \mathcal{P}(B) = \emptyset$

for if $X \subseteq A$, then any $x \in X$ lies in A & therefore not in B, so

$$X \cap B = \emptyset \quad \forall X \in \mathcal{P}(A)$$

& similarly $A \cap Y = \emptyset$, $\forall Y \in \mathcal{P}(B)$

Therefore $X \cap Y = \emptyset$ for all $X \in \mathcal{P}(A)$, $Y \in \mathcal{P}(B)$, and we conclude that

$$\mathcal{P}(A) \cap \mathcal{P}(B) = \emptyset$$

This means

$$\mathcal{P}(A) - \mathcal{P}(B) = \emptyset$$

and since $A - B = A$,

$$\mathcal{P}(A-B) = \mathcal{P}(A)$$

$$\begin{aligned} \Longrightarrow \quad \mathcal{P}(A) - \mathcal{P}(B) &= \emptyset \\ &\subseteq \mathcal{P}(A) \\ &= \mathcal{P}(A-B) \end{aligned}$$

(14) $\mathcal{P}(A) \cap \mathcal{P}(B) = \mathcal{P}(A \cap B)$ is true!

pf: $X \in \mathcal{P}(A) \cap \mathcal{P}(B) \Rightarrow X \subseteq A \ \& \ X \subseteq B$

$\Rightarrow X \subseteq A \cap B$

$(x \in X \Rightarrow x \in A \ \& \ x \in B)$

$\Rightarrow X \in \mathcal{P}(A \cap B)$

Conversely, if $X \in \mathcal{P}(A \cap B)$ then $X \subseteq A \cap B \subseteq A$

$\Rightarrow X \subseteq A$

and similarly $X \subseteq A \cap B \subseteq B \Rightarrow X \subseteq B$

so

$X \in \mathcal{P}(A) \cap \mathcal{P}(B)$. $\boxtimes$

(24) $2^x \geq x+1 \ \forall x \in (0, \infty)$ is false

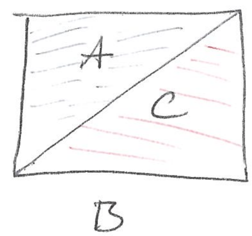
ex. $x = \frac{1}{2}$ gives

$\sqrt{2} < \frac{3}{2} \iff 2 < \frac{9}{4}$

which is true

(26) $A = B - C \Rightarrow B = A \cup C$ is False

Ex.



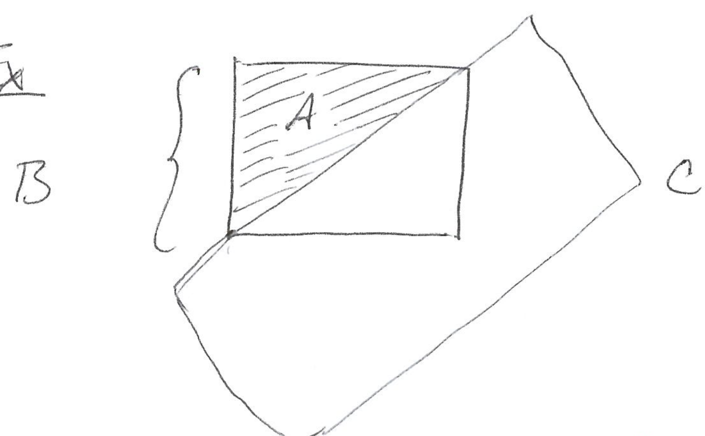
$A = B - C$

&

$B = A \cup C$

but

Ex



$A = B - C \not\Rightarrow B = A \cup C$

$(B \subsetneq A \cup C)$

(28) $a, b \in \mathbb{Z}$, $a | b$ & $b | a \Rightarrow a = b$ is False

ex. $a = 1$, $b = -1$