HW 6 Evens

Chapter 4

(2) Let $x \in \mathbb{Z}$ be odd, so $x = 2k+1$, for some $k \in \mathbb{Z}$, and compute x^3:

$$
\begin{aligned}
x^3 &= (2k+1)^3 \\
&= (2k)^3 + 3(2k)^2 + 3(2k) + 1 \quad \text{(Binomial Thm.)} \\
&= 2\underbrace{(4k^3+6k^2+3k)}_{=: \ell \in \mathbb{Z}} + 1 \\
&= 2\ell + 1
\end{aligned}
$$

so x^3 is odd.

(6) Let $a, b, c \in \mathbb{Z}$ and suppose $a|b$ & $a|c$.

Then $ak = b$ and $al = c$ for some $k, l \in \mathbb{Z}$

so

$$
\begin{aligned}
b + c &= ak + al \\
&= a(k+l) = am \quad \text{where } m = k + l \in \mathbb{Z}
\end{aligned}
$$

$\Rightarrow a | (b+c)$

(8) Let $a \in \mathbb{Z}$ and suppose $5 | 2a$. This means

$$2a = 5k \text{ for some } k \in \mathbb{Z} \qquad (*)$$

This shows that $5k$ is even, so 2 either divides 5 or k. Since $2 \nmid 5$, we must have $2|k$, so

$$k = 2l \text{ for some } l \in \mathbb{Z} \qquad (**)$$

Plugging this back into $(*)$, we get

$$\begin{aligned} \cancel{2}a &= 5k \\ &= 5(2l) \\ &= \cancel{2}(5l) \end{aligned}$$

(cancellation law)

$\implies a = 5l$

$\implies 5 | a$

QED

(12) Let $x \in \mathbb{R}$, $0 < x < 4$. By Axiom 3.1 of $\mathbb{R}$ we know

$$0 = x - x < 4 - x$$

FYI. But more importantly, Thm. 5.6.2 of $\mathbb{R}$ says

$$(4-x) > 0 \ \& \ x > 0$$

$$\Longrightarrow \quad x(4-x) > 0$$

and even more importantly, Thm. 5.6.6 says

$$(x-2)^2 \geq 0$$

$$\left.\begin{aligned} &\Longrightarrow \quad x^2 - 4x + 4 \geq 0 \\ &\Longrightarrow \quad x(x-4) + 4 \geq 0 \end{aligned}\right\} \text{axioms \& arithmetic properties of } \mathbb{R}$$

$$\Longrightarrow \quad 4 \geq -x(x-4) \qquad \text{(Axiom 3.2, } \mathbb{R}\text{)}$$
$$= x(4-x)$$

$$\Longrightarrow \quad \frac{4}{x(x-4)} \geq 1$$

QED

(14) Let $n \in \mathbb{Z}$ and consider $5n^2+3n+7$.

Case 1: $n=2k$ is even $\Longrightarrow$

$$\begin{aligned} 5n^2+3n+7 &= 5(2k)^2+3(2k)+6+1 \\ &= 2\underbrace{(10k^2+3k+3)}_{=l\in\mathbb{Z}}+1 \\ &= 2l+1 \end{aligned}$$

is odd.

Case 2: $n=2k+1$ is odd $\Longrightarrow$

$$\begin{aligned} 5n^2+3n+7 &= 5(2k+1)^2+3(2k+1)+7 \\ &= 5(4k^2+4k+1)+6k+10 \\ &= 20k^2+26k+15 \\ &= 2\underbrace{(10k^2+13k+7)}_{=l\in\mathbb{Z}}+1 \\ &= 2l+1 \end{aligned}$$

is odd. So in all cases $5n^2+3n+7$ is odd.

(18) Let $x, y \in \mathbb{R}$ satisfy

$$\left.\begin{array}{l} x, y > 0 \\ x < y \end{array}\right\} \quad \text{i.e.} \quad 0 < x < y$$

Then by Axiom 3.1 of $\mathbb{R}$ we have

$$0 = x - x < y - x$$

& Axiom 3.2 then gives

$$\begin{aligned} 0 &< x(y-x) \\ &= xy - x^2 \end{aligned}$$

$$\Longrightarrow \text{(Axiom 3.1)} \qquad x^2 < xy \qquad (1)$$

Similarly,

$$0 < y(y-x) = y^2 - xy$$

$$\Longrightarrow \qquad xy < y^2 \qquad (2)$$

Combining (1) & (2): $\quad x^2 < xy < y^2$

$$\Longrightarrow \quad x^2 < y^2 \text{ by transitivity of "<"}$$

QED

(20) Let $a \in \mathbb{Z}$ and suppose $a^2 | a$. By def., this means

$$a = a^2 k \text{ for some } k \in \mathbb{Z}$$

case 1 : $\Rightarrow 1 = ak$ (cancellation law)
($a \neq 0$)

$\Rightarrow a \neq 0$ (Lemma 5.4 of TR)

$\Rightarrow a = k = \pm 1$

case 2
($a = 0$)

$a = 0 \Rightarrow a^2 = a = a \cdot 1$

$\Rightarrow a^2 | a$ since $1 \in \mathbb{Z}$.

(22) Let $n \in \mathbb{N}$, & consider cases :

case 1 : $\binom{1}{2} \overset{\text{def}}{=} 0 \Rightarrow 2\binom{1}{2} + \binom{1}{1}$
($n = 1$)

$n=1$

$= 0 + 1$

$= 1^2$

case 2
($n \geq 2$)

$$2\binom{n}{2} + \binom{n}{1} = \cancel{2} \cdot \frac{n!}{\cancel{2}!\,(n-2)!} + \frac{n!}{1!\,(n-1)!}$$

$$= \frac{n(n-1)\cdot\cancel{(n-2)!}}{\cancel{(n-2)!}} + \frac{n\,\cancel{(n-1)!}}{\cancel{(n-1)!}}$$

$$= n(n-1) + n$$

$$= n^2 - n + n = n^2$$

(26) Let $n = 2k+1$, $a = k+1$, $b = k$:

$$\begin{aligned} a^2 - b^2 &= (k+1)^2 - k^2 \\ &= \cancel{k^2} + 2k + 1 - \cancel{k^2} \\ &= 2k+1 \\ &= n \end{aligned}$$

Ch. 5 Evens

(2) Let $n \in \mathbb{Z}$ and suppose n is odd ($\sim Q$, in $P \to Q$). Then

$$n = 2k+1 \quad \text{for some } k \in \mathbb{Z}$$

$$\Longrightarrow n^2 = (2k+1)^2$$

$$= 4k^2 + 4k + 1$$

$$= 2\underbrace{(2k^2+2k)}_{= l \in \mathbb{Z}} + 1$$

$$= 2l + 1$$

is odd. This is the contrapositive of

n^2 even $\Rightarrow$ n even.

(8) Let $x \in \mathbb{R}$ & suppose $x < 0$ $(\sim(x \geq 0))$.

Then,

$$x^5 - 4x^4 + 3x^3 - x^2 + 3x - 4$$

$$= \underbrace{\underbrace{x^4}_{\geq 0}\underbrace{(x-4)}_{<0}}_{<0} + \underbrace{\underbrace{x^2}_{\geq 0}\underbrace{(3x-1)}_{<0}}_{<0} + \underbrace{3x-4}_{<0}$$

$$< 0$$

bec. $x < 0 \Rightarrow x - 4 < -4 < 0$

$\Rightarrow x - 4 < 0$

& $x < 0 \Rightarrow 3x < 0$

$\Rightarrow 3x - 1 < -1 < 0$

$\Rightarrow 3x - 1 < 0$ & sim. $3x - 4 < 0$

while $x^4, x^2 \geq 0 \Rightarrow x^4(x-4) < 0$

& $x^2(3x-1) < 0$

Axiom 3.2 of $\mathbb{R}$

QED

(10) $\underbrace{x,y,z \in \mathbb{Z}$ & suppose $x \mid y \text{ or } x \mid z}_{= \sim(x \nmid y \text{ and } x \nmid z)}$

Then,

$$y = kx \quad \text{or} \quad z = lx$$

for some $k, l \in \mathbb{Z}$. Say $y = kx$.

Then,

$$\begin{aligned} yz &= (kx)z \quad \Rightarrow \quad x \mid yz \\ &= \underbrace{(kz)}_{\in \mathbb{Z}} x \end{aligned}$$

(12) Let $a \in \mathbb{Z}$ be even, $a = 2k$. Then,

$$a^2 = (2k)^2 = 4k^2 \Rightarrow 4 \mid a^2$$

(20) $a \in \mathbb{Z}$ & $a \equiv 1 \bmod 5 \Rightarrow \underbrace{a-1 = k5}$ plug in

$$\begin{aligned} &\Rightarrow a^2 - 1 = (a-1)(a+1) \\ &\qquad\qquad = k \cdot \underbrace{5(a+1)}_{\in \mathbb{Z}} \\ &\Rightarrow a^2 \equiv 1 \bmod 5. \end{aligned}$$

oops, should've done (20) only. Sorry!

(21) $a, b \in \mathbb{Z}$, $n \in \mathbb{N}$. Then

$$a \equiv b \bmod n \implies a^3 \equiv b^3 \bmod n$$

i.e. $a = b \in \mathbb{Z}_n \implies a^3 = b^3 \in \mathbb{Z}_n$

pf: $a \equiv b \bmod n$

$$\implies a - b = kn$$

$$\implies a^3 - b^3 = (a-b)(a^{n-1} + a^{n-2}b + \cdots + b^{n-1})$$

$$= kn(\quad " \quad " \quad)$$

$$\implies a^3 \equiv b^3 \bmod n \text{ or } a^3 \equiv b^3 \text{ in } \mathbb{Z}_n$$

(24) $a \equiv b \bmod n$, $c \equiv d \bmod n$

$$\implies ac \equiv bd \bmod n$$

pf: $a - b = kn$, $c - d = ln \implies a = kn + b$
$c = ln + d$

$$\implies ac - bd = (kn+b)(ln+d) - bd$$

$$= (kln)n + (kd+bl)n + \cancel{bd} - \cancel{bd}$$

$$= (kln + kd + bl)n$$

$$\implies ac \equiv bd \bmod n$$

(26) If $k \in \mathbb{N}$ (i.e. $k \geq 1$, according to Hammack, p. 4) and $n = 2^k - 1$, then every entry in row n of Pascal's Triangle is odd, i.e. every binomial coefficient

$$\binom{n}{l} = \frac{n!}{l!\,(n-l)!}$$

is odd, $l = 0, 1, \ldots, n$.

pf: First, note that

$$n = 2^k - 1 = 2^k - 2 + 1$$
$$= 2(2^{k-1} - 1) + 1$$

is odd, so there are ($l = 0, 1, \ldots, n$) $n+1$ coefficients $\binom{n}{l}$ to consider, an <u>even number</u> of coefficients. We know, moreover, that

$$\left.\begin{array}{l} \binom{n}{n} = \binom{n}{0} = 1 \\ \text{and, for the record, } \binom{n}{n-1} = \binom{n}{1} = n \end{array}\right\} \text{all odd.}$$

so it remains to check $\binom{n}{\ell}$ for $\ell = 2, \dots, n-2$.

Suppose, therefore, that

$$2 \le \ell \le n-2$$

Now,

$$n! = (2^k-1)!$$

$$= (2^k-1)(2^k-2)(2^k-3)\cdots(2^k-\ell)\,(2^k-1-\ell)!$$

so

$$\binom{n}{\ell} = \frac{n!}{\ell!\,(n-\ell)!}$$

$$= \frac{(2^k-1)(2^k-2)\cdots(2^k-\ell)\cdot\overbrace{\cancel{(2^k-1-\ell)!}}^{=(n-\ell)!}}{\ell!\,\cancel{(n-\ell)!}}$$

$$= \frac{(2^k-\underline{1})(2^k-\underline{\underline{2}})\cdots(2^k-\ell)}{\ell\cdot(\ell-1)\cdots 3\cdot\underline{\underline{2}}\cdot\underline{1}}$$

An observation is now in order:

odd

$$z^k - 1 = z^k - 1^k \leftarrow a^k - b^k = (a-b)(a^{k-1} + a^{k-2}b + \cdots + b^{k-1})$$

$$= \underbrace{(z-1)}_{=1}\underbrace{(z^{k-1} + z^{k-2} + \cdots + 1)}_{= z(z^{k-2} + \cdots + z + 1) + 1}$$

$$= z^{k-1} + z^{k-2} + \cdots + z^2 + z + 1$$

even

$$z^k - \underline{z} = z(z^{k-1} - 1)$$

$$= z((z-1)(z^{k-2} + z^{k-3} + \cdots + z + 1))$$

$$= \underbrace{z}_{z}\underbrace{(z^{k-2} + z^{k-3} + \cdots + 1)}_{\text{odd}}$$

odd

$$z^k - 3 = z^k - z - 1$$

$$= z(z^{k-2} + z^{k-3} + \cdots + z + 1) - 1$$

even

$$z^k - \underline{4} = z^2(z^{k-2} - 1)$$

$$= \underbrace{z^2}_{z^2}\underbrace{(z^{k-3} + z^{k-4} + \cdots + z + 1)}_{\text{odd}}$$

etc. Thus, $z^k - r$, when r is even, allows us to factor out of $z^k - r$ all powers of z in r!

Back to $\binom{n}{l}$: we said above that

$$\binom{n}{l} = \frac{(2^k-1)(2^k-2)(2^k-3)\cdots(2^k-l)}{1\cdot 2\cdot 3\cdots l}$$

factor out of the evens & cancel $\longrightarrow$

$$= \frac{(2^k-1)\cdot\cancel{2}(2^{k-1}-1)(2^k-3)\cdot\cancel{2^2}(2^{k-2}-1)\cdots}{1\cdot\cancel{2}\cdot 3\cdot\cancel{4}\cdots l}$$

$$= \frac{\text{odd}\cdot\text{odd}\cdots\text{odd}}{\text{odd}\cdot\text{odd}\cdots\text{odd}}$$

$$= \text{an odd \#} \quad \text{(see below)}$$

By our observation on the previous page, all powers of 2 factored out upstairs cancel exactly w/all powers of 2 downstairs, so we are left only with odds. Products of odds are odd (Ch. 4 #4), so both numerator & denom. are odd.

Now, for the punch line: if $\binom{n}{l}$ were even, say $\binom{n}{l} = 2m$, then $(\text{odd\#})\cdot 2m = (\text{odd\#})$ <u>Impossible</u>!