HW 5 – Evens

Sec. 3.2

(6) 2 ways to land a coin
6 ways to land a die
52 ways to draw a card

$\Rightarrow$ $\boxed{2 \cdot 6 \cdot 52 = 624 \text{ possible outcomes}}$

If the die is to land on 3, there are

$\boxed{2 \cdot 1 \cdot 52 = 104 \text{ ways/outcomes}}$

If the die is to land on an odd #, there are

$\boxed{2 \cdot 3 \cdot 52 = 312 \text{ outcomes}}$

3 odd #s,
1, 3, 5

If the die is to land on an odd & the card is to be a king, there are

$\boxed{2 \cdot 3 \cdot 4 = 24 \text{ ways}}$

there are
4 kings

(8) 10 tosses means length-10 strings, each term of which is H or T:

$$\Rightarrow \boxed{\underbrace{2\cdot 2\cdots 2}_{\substack{10\ \text{slots, 2} \\ \text{choices each}}} = 2^{10} = \underset{\text{possible strings}}{1024}}$$

(10) $\boxed{6\cdot 6\cdot 6\cdot 6 = 6^4 = 1269 \text{ poss. outcomes}}$

Sec. 3.3

(2) $\cancel{P(14,5) = \frac{14!}{(14-5)!} = \frac{14!}{9!} = \boxed{240{,}240 \text{ ways}}}$

-1

This should be 4*P(13,5) = 617,760 ways, and not P(14,5). I made two mistakes here!

(8) (a) 3 vowels, A, E, I (no repetition, 1st letter a vowel)

7 consonants, B, C, D, F, G, H, J

	9	8	7	6	
3	~~7~~	~~6~~	~~5~~	~~4~~	
①	②	③	④	⑤	(slot #'s)

3 vowels

I must have been tired that night. Why did I use the consonants instead of the rest of the letters in (a) and (b)?

so $3 \cdot 7 \cdot 6 \cdot 5 \cdot 4 = 3 \cdot \frac{7!}{(7-4)!} = \cancel{\boxed{\begin{array}{c}2520\\ \text{ways}\end{array}}}$ = 9072

(b)

~~7 consonants to start~~

	8	7	6		
3	~~7~~	~~6~~	~~5~~	2	
①	②	③	④	⑤	(slot #'s)

3 vowels to start

2 vowels left after 1st was chosen

$\Longrightarrow 3 \cdot 2 \cdot 7 \cdot 6 \cdot 5 = \frac{3!}{(3-2)!} \cdot \frac{7!}{(7-3)!}$

or

$P(3,2) \cdot P(7,3)$

$= \cancel{\boxed{1260 \text{ ways}}}$

= 2016

-1

(c) no repetitions, exactly 1 A:

There are 5 slots from which to choose for A. Once that has been fixed, there are 4 slots left in which to permute the remaining 9 letters:

$$5 \cdot P(9,4) = 5 \cdot \frac{9!}{(9-4)!} = \boxed{15120 \text{ ways}}$$

(10) Repetition allowed, P, R, O, F, S & last letter is S, length 6, O used ≥ 2 times.

fixed

$\_\ \ \_\ \ \_\ \ \_\ \ \_\ \ \boxed{S}$

1 2 3 4 5 6

There remain 5 slots, 2 of which are to be filled with O's, & there are $\binom{5}{2} = 10$ ways to do this.

there remain 3 slots in which we can put any of the 6 letters:

$$\binom{5}{2} \cdot 6^3 = 10 \cdot 6^3 = \boxed{2160 \text{ ways}}$$

Sec. 3.4

(2) For which n does $n!$ have $\leq n$ digits?

A: $\boxed{n = 1, \ldots, 24. \quad 25! \text{ has } 26 \text{ digits while } 24! \text{ has } 24.}$

In fact, here's how I reckoned: after wasting some time looking at 1! through 6! = 720, I decided ~~[illegible]~~ to look at

$9! = 362880$ has 6 digits < 9

and after that we multiply by 2-digit numbers: 10, 11, ..., so we gain a digit, maybe two each time; so I decided to take samples:

$15! = 1{,}307{,}674{,}368{,}000$ has 13 digits

$20! = 2{,}432{,}902{,}008{,}176{,}640{,}000$

has 19 digits

Getting close!

$25! = 15{,}511{,}210{,}043{,}330{,}985{,}984{,}000{,}000$

has 26 digits! Success!

$25 = n > 26 = \#$ digits

Then I backtracked, & found that 24! has 24 digits. Clearly 23 & below will only lose digits one or more at a time, so we are done.

(4) $\dfrac{100!}{95!} = \dfrac{100\cdot 99\cdot 98\cdot 97\cdot 96\cdot \cancel{95!}}{\cancel{95!}}$

$= 100\cdot 99 \cdots 96$

$= 9{,}034{,}502{,}400$ (I used Wolfram to compute this product).
(at first)

$*\ 100\cdot 99\cdots 96 = 100(100-1)(100-2)(100-3)(100-4)$

by hand:

$$= 100^5 + \underbrace{(-1)100^4 + (-2)100^4 \cdots + (-4)100^4}$$

$$= (-1-2-3-4)100^4$$

$$= (-10)\cdot 100^4$$

$$+ \underbrace{(-1)(-2)10^3 + (-1)(-3)10^3 + \cdots}$$

$$= \big[(-1)(-2) + (-1)(-3) + (-1)(-4) + (-2)(-3) + (-2)(-4) + (-3)(-4)\big]100^3$$

$$= 35\cdot 100^3$$

$$+ \underbrace{(-1)(-2)(-3)10^2 + (-1)(-2)(-4)10^2 + \cdots}$$

$$= \big[(-1)(-2)(-3) + (-1)(-2)(-4) + (-1)(-3)(-4) + (-2)(-3)(-4)\big]$$

$$= [-6-8-12-24]\cdot 100^2$$

$$= (-50)\cdot 100^2$$

100^2

$$+\underbrace{(-1)(-2)(-3)(-4)\,100}_{=\,24\cdot 100}$$

$$= 100^5 - 10\cdot 100^4 + \overset{35}{\cancel{25}}\cdot 100^3 - 50\cdot 100^2 + 24\cdot 100$$

$$= 100\left(100^4 - 10\cdot 100^3 + \overset{3}{\cancel{2}}5\cdot 100^2 - 50\cdot 100 + 24\right)$$

$$= 100\left(\begin{array}{r} \left.\begin{array}{r} 100{,}000{,}000 \\ -10{,}000{,}000 \end{array}\right\} = 90{,}000{,}000 \\ \left.\begin{array}{r} +\ \overset{3}{\cancel{2}}50{,}000 \\ -\ 5{,}000 \end{array}\right\} = 345{,}000 \\ +\ 24 \\ \hline \end{array}\right)$$

$$= 9{,}034{,}502{,}400$$

(6) Let's first study 10! & understand it:

$$10! = \underline{\underline{10}}\cdot 9\cdot 8\cdot 7\cdot 6\cdot \underline{\underline{5}}\cdot 4\cdot 3\cdot \underline{\underline{2}}\cdot 1$$

another 10

$$= \underbrace{10^2}\cdot\underbrace{(9\cdot 8\cdot 7\cdot 6\cdot 4\cdot 3\cdot 2)}$$

no other 5 in here, so no 10 in here

only 2 10's

$$= \underbrace{\text{stuff}}_{\text{the prod. of } 9\cdot 8\cdot 7\cdot 6\cdot 4\cdot 3\cdot 2}\cdot 00$$ only 2 0's at end.

2 0's; one 0

Now, $100! = \underline{\underline{100}}\cdot 99\cdot 98\cdot 97\cdot 96\cdot \underline{\underline{95}}\cdot 94\cdot 93\cdot \underline{\underline{92}}\cdot 91$

one 0 $\cdot\underline{\underline{90}}\cdot 89\cdot \ldots\ \cdot\underline{\underline{85}}\cdot \ldots\ \underline{\underline{82}}\cdot 81$ one 0

$\vdots$

$\underline{\underline{10}}\cdot 9 \ldots\ \underline{\underline{5}}\ \ldots\ \underline{\underline{2}}\cdot 1$

= 3 0's fr. row 1

+ 2 0's fr. rows other 9 rows } after product

$= 3 + 9\cdot 2 = \boxed{21 \text{ 0's}}$ after prod.

$$* \ 95\cdot 92$$
$$= (90+5)(90+2)$$
$$= 9^2\cdot 10^2 + (2+5)\cdot 90 + 10$$
$$= 10(9^2\cdot 10 + 7\cdot 9 + 1)$$
$$= 10(810 + 64)$$
$$= 8740$$

But I forgot that the rows with 25, with 50=25*2, and with 75=25*3 have *two* 5's, which get multiplied with 2's each from any of the evens in that row. So add 3 to 21 to get $\boxed{24.}$

(8) <u>odd digit possibilities</u>: 1, 3, 5, 7 may be permuted in $4! = 24$ ways,

<u>even digit possibilities</u>: 2, 4, 6 may be permuted in $3!$ ways, but we have to also consider those odds:

- <u>odds</u> followed by evens (e.g. 1,3,5,7, 2,4,6)
- even, odds, 2 evens (e.g. 2, 1,3,5,7, 4,6)
- 2 evens, odds, even (e.g. 2,4, 1,3,5,7, 6)
- evens followed by odds (e.g. 2,4,6, 1,3,5,7)

Each of these 4 has $4! \cdot 3! = 24 \cdot 6 = 144$ possibilities, so in total we have

$$4 \cdot 144 = \boxed{576 \text{ ways}}$$

(12) There are 13 clubs, so 39 non-clubs, in a deck, while there are

$$P(39,7) = \frac{39!}{(39-7)!} = \frac{39!}{32!}$$

$$= 77,519,992,480$$

ways to permute 7 of them.

(18) Let $\Gamma(x) \stackrel{\text{def}}{=} \int_0^\infty t^{x-1} e^{-t}\,dt$.

$$\Gamma(1) = \int_0^\infty t^{1-1} e^{-t}\,dt$$

$$= \int_0^\infty e^{-t}\,dt$$

$$= \lim_{b\to\infty} \int_0^b e^{-t}$$

$$= \lim_{b\to\infty} -e^{-t}\Big|_0^b$$

$$= \lim_{b\to\infty} 1-e^{-b}$$

$$= 1$$

$$\Gamma(2) = \int_0^\infty t^{2-1} e^{-t}\,dt$$

$$= \int_0^\infty t e^{-t}\,dt$$

$$\begin{pmatrix} \text{by parts:} & \\ u = t & v = -e^{-t} \\ u' = 1 & v' = e^{-t} \end{pmatrix}$$

$$= \lim_{b\to\infty} \Big(\underbrace{\left[-te^{-t}\right]_0^b}_{\substack{= -be^{-b} \\ = -\frac{b}{e^b} \to 0 \\ \text{by L'Hopitals}}} + \underbrace{\int_0^b e^{-t}\,dt}_{\substack{\to 1 \text{ as above} \\ = \Gamma(1)}} \Big)$$

$$= 1$$

$$\Gamma(3) = \int_0^\infty t^{3-1} e^{-t}\,dt$$

$$= \int_0^\infty t^2 e^{-t}\,dt \qquad \begin{pmatrix} u = t^2 & v = -e^{-t} \\ u' = 2t & v' = e^{-t} \end{pmatrix}$$

$$= \underbrace{\left[-t^2 e^{-t}\right]_0^\infty}_{= 0 \text{ by L'H}} + 2\underbrace{\int_0^\infty t e^{-t}\,dt}_{= \Gamma(2) = 1}$$

$$= 2$$

$$\Gamma(4) = \int_0^\infty t^{4-1} e^{-t}\,dt = \int_0^\infty t^3 e^{-t}\,dt \qquad \begin{pmatrix} u = t^3 & v = -e^{-t} \\ u' = 3t^2 & v' = e^t \end{pmatrix}$$

$$= \underbrace{\left[-t^3 e^{-t}\right]_0^\infty}_{\substack{=0 \text{ by} \\ L'H}} + 3\underbrace{\int_0^\infty t^2 e^{-t}\,dt}_{=\Gamma(3)=2}$$

$$= 3 \cdot 2$$

Thus,

$$\boxed{\begin{aligned} \Gamma(1) &= 1 = 0! \\ \Gamma(2) &= 1 = 1! \\ \Gamma(3) &= 2 = 2! \\ \Gamma(4) &= 3\cdot 2 = 3! \end{aligned}}$$

by induction

In general, since $\Gamma(1) = 1 = (1-1)!$, if $\Gamma(n-1) = (n-2)!$ then

$$\boxed{\begin{aligned} \Gamma(n) &= \int_0^\infty t^{n-1} e^{-t}\,dt = \underbrace{\left[-t^{n-1} e^{-t}\right]_0^\infty}_{=0 \text{ by } L'H} + (n-1)\underbrace{\int_0^\infty t^{n-2} e^{-t}\,dt}_{\substack{=\Gamma(n-1) \\ =(n-2)!}} \\ &= (n-1)(n-2)! = (n-1)! \end{aligned}}$$

Section 3.5

(2) $|A| = 100 \implies \binom{100}{5} = 75{,}287{,}520$

subsets of A have size 5.

(4) $|\{X \mid X \in \mathcal{P}(B),\ |X| = 6\}| = 28$

$$\implies \binom{n}{6} = 28$$

$$\implies \frac{n!}{6!\,(n-6)!} = 28$$

Well, $\binom{7}{6} = 7$, $\binom{8}{6} = \dfrac{8\cdot 7\cdot \cancel{6!}}{\cancel{6!}\,2!} = 28$

$$\implies \boxed{n = 8}$$

(14) $0 \le k \le n \implies \binom{n}{k} = \dfrac{n!}{k!\,(n-k)!}$

$$= \frac{n!}{(n-k)!\,\underbrace{(n-(n-k))}_{\substack{n-(n-k)\\ = n-n+k = k}}!} = \binom{n}{n-k}$$

(18) $\binom{10}{0} = 1$ has 0 "1"'s

$\binom{10}{2} = \frac{10 \cdot 9}{2} = 45$ have 2 "1"'s

$\binom{10}{4} = \frac{10 \cdot \cancel{9}^{3} \cdot \cancel{8} \cdot 7}{\cancel{4} \cdot \cancel{3} \cdot \cancel{2} \cdot 1} = 210$ have 4 1's

$\binom{10}{6} = \binom{10}{4} = 210$ have 6 1's

$\binom{10}{8} = \binom{10}{2} = 45$ have 8 1s

$+ \quad \binom{10}{10} = 1$ has 10 1's

$= 1 + 2 \cdot 45 + 2 \cdot 210 + 1$

$= \boxed{512}$

Section 3.6

(2) $\binom{13}{8} = \boxed{1287}$

(6) $$2^n = (1+1)^n = \sum_{k=0}^{n} \binom{n}{k} 1^{n-k} \cdot 1^k$$

$$= \sum_{k=0}^{n} \binom{n}{k}$$

(12) $$\binom{n}{k}\binom{k}{m} = \frac{n!}{\cancel{k!}\,(n-k)!} \cdot \frac{\cancel{k!}}{m!\,(k-m)!}$$

while

$$\binom{n}{m}\binom{n-m}{k-m} = \frac{n!}{m!\,\cancel{(n-m)!}} \cdot \frac{\cancel{(n-m)!}}{(k-m)!\,\underbrace{(n-m-(k-m))!}_{=(n-k)!}}$$

So they are equal.