MATH 2300 - Homework 5

Instructions: This homework is due on Friday, July 5^{th}. You may work with other students, however each person is responsible for writing their own solutions. Please write the names of any students who helped you.

1. (a) Show that $1 = 0.999\ldots.$

   (b) Find an irreducible rational number $\dfrac{p}{q}$ such that $\dfrac{p}{q} = 0.13571357\ldots.$

2. Determine if the following converge or diverge:

   (a) $\displaystyle\sum_{k=1}^{\infty}\left(\frac{1}{5^k} - \frac{1}{k(k+1)}\right)$

   (b) $\displaystyle\sum_{n=1}^{\infty}\left(1 + \frac{1}{k}\right)^k$

3. Find the sum of the following series:

   (a) $\displaystyle\sum_{k=1}^{\infty}\left(\frac{1}{5^k} - \frac{1}{k(k+1)}\right)$

4. The following problem will highlight the use of what is called the **geometric mean**.

   (a) If inside of a right triangle, a line segment perpendicular to the hypotenuese that passes through the right angle is drawn, the length of this line segment can be found using the fact that the two inside triangles that were formed are similar. Suppose that the line segement drawn seperates the hypotenuese into a segment of length a and a segment of length b. Find the length of the line segment.
   Note: For some a and b, the ratio of $a+b$ to a is the same as the ratio of a to b. When this occurs, the ratio of a to b is called the golden ratio.

   (b) The geometric mean, g, of n numbers is the n^{th} root of their product

   $$g = \sqrt[n]{x_1 x_2 \cdots x_n}.$$

   Find the geometric mean of the first and third elements of the sequence $a_n = 3(\frac{1}{5})^n$. Let n start at 0.

   (c) Show that for all $k < n$ the geometric mean of the $n-k^{\text{th}}$ and $n+k^{\text{th}}$ term of a geometric series is the n^{th} term.

   (d) Suppose the first term of the sequence b_n is 8.1 and the fifth term is 240.1. Find the first 5 terms.

   (e) Suppose c_n is a geometric series with $c_0 = 15$ and $c_2 = \frac{5}{3}$. Write an explicit formula for the n^{th} term.

5. Let S be the sun of a convergent series $\displaystyle\sum_{n=1}^{\infty} u_n$ where $u_n > 0$ for all n. Let S_N be the N^{th} partial sum of the series, and let $E_N = S - S_N$ be the error in the N^{th} partial sum approximation. Suppose also that for some integrable function $f(x)$, $f(n) = u_n$ for all n.

(a) Show that $S - S_N < \int_N^\infty f(x)dx$ if $f'(x) < 0$ for all $x > N$.

(b) Use part (a) to give a lower bound for n such that $\sum_{k=1}^{n} \frac{1}{k^4}$ approximates $\sum_{k=1}^{\infty} \frac{1}{k^4}$ with $E_n < 10^{-5}$.

(c) Give an upper bound on the error when $\sum_{k=1}^{\infty} \frac{1}{k \ln(k)}$ is approximated by S_10.

6. For which positive values of α does $\sum_{k=1}^{\infty} \frac{\alpha^k}{k^\alpha}$ converge?

7. Classify the following series as absolutely convergent, conditionally convergent, or divergent.

(a) $\sum_{k=1}^{\infty} (-1)^{k+1} \left(\frac{k+2}{3k-1} \right)^k$

(b) $\sum_{k=1}^{\infty} \cos(n\pi) n e^{-n^2}$

(c) $\sum_{n=1}^{\infty} (-1)^k \frac{k}{5^k}$

(d) $\sum_{n=1}^{\infty} (-1)^{k+1} \frac{k^k}{k!}$

8. Find a series for which $\sum (a_n)^2$ converges but $\sum |a_n|$ diverges.