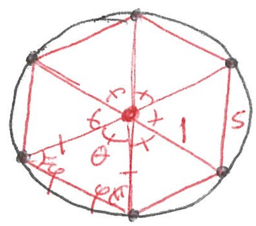
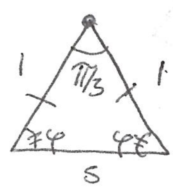
# HW 4 Evens & (1)–(3) (Hammack)

(1) We start with the hexagon:



Since $\theta = \frac{2\pi}{6} = \pi/3$ and each subtriangle is isosceles,



we must have

$$2\varphi + \frac{\pi}{3} = \pi$$

$$\Rightarrow \quad 2\varphi = \pi - \pi/3 = \frac{2\pi}{3}$$

$$\Rightarrow \quad \varphi = \frac{\pi}{3}$$

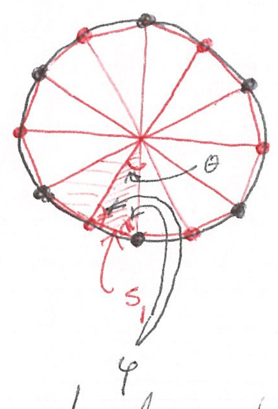
so the triangle is an <u>equilateral</u> triangle, and

$$\boxed{s = 1}$$

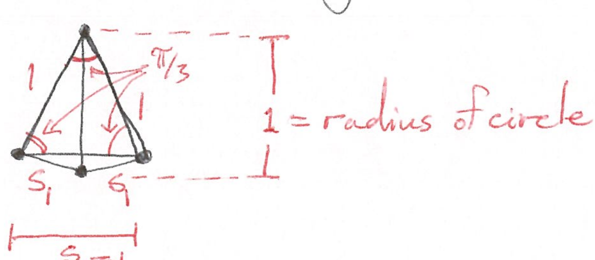
Let us define the first term of our recursive sequence,

$$s_0 \overset{\text{def}}{=} 1$$

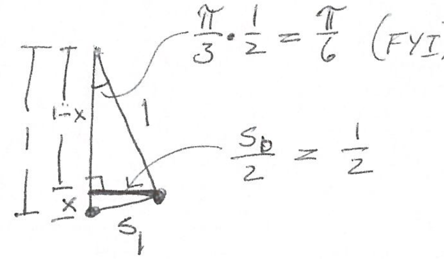
Now let's double the number of sides and find the new sidelength $s_1$ = sidelength of the regular 12-gon inscribed in the circle:



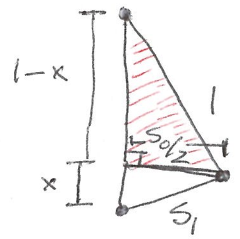

In the pair of adjecent triangles




we have some prior information:



Let's use Pythagorean Thm. twice:




$$(1-x)^2+\left(\frac{s_0}{2}\right)^2=1^2$$

$$\Rightarrow \cancel{1}-2x+x^2+\frac{s_0^2}{4}=\cancel{1}$$

$$\Rightarrow x^2-2x+\frac{s_0^2}{4}=0$$

$$\Rightarrow x=\frac{2\pm\sqrt{2^2-4\cdot1\cdot\frac{s_0^2}{4}}}{2\cdot1}$$

$$=\frac{2\pm\sqrt{4-s_0^2}}{2}$$

$$=\frac{2}{2}\pm\frac{\sqrt{4-s_0^2}}{2}$$

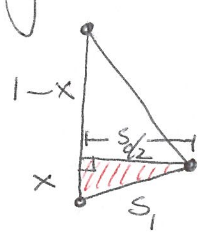
$$=1\pm\sqrt{\frac{4}{4}-\frac{s_0^2}{4}}$$

$$=1\pm\sqrt{1-\frac{s_0^2}{4}}$$

We toss the plus, since $0<x<1$ & adding the root would make $x>1$. Thus,

$$x=1-\sqrt{1-\frac{s_0^2}{4}}$$

Pythagorean Thm. on the other triangle gives: plug in




$$x^2+\frac{s_0^2}{4}=s_1^2$$

$$\Rightarrow s_1=\sqrt{x^2+\frac{s_0^2}{4}}=\sqrt{\left(1-\sqrt{1-s_0^2/4}\right)^2+s_0^2/4}$$

$$= \sqrt{\left(1 - 2\sqrt{1 - s_0^2/4} + \left(\sqrt{1 - s_0^2/4}\right)^2\right) + s_0^2/4}$$

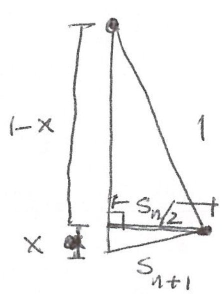
$$= \sqrt{1 - 2\sqrt{1 - s_0^2/4} + 1 - \cancel{s_0^2/4} + \cancel{s_0^2/4}}$$

$$= \sqrt{2 - 2\sqrt{1 - s_0^2/4}}$$

But what worked for $s_0$ & $s_1$, namely

$$s_1 = \sqrt{2 - 2\sqrt{1 - s_0^2/4}}$$

could work equally well on $s_{n+1}$ and $s_n$, by the same means:

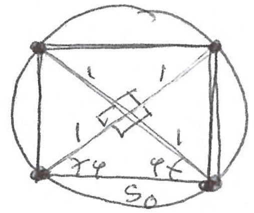



$$(1-x)^2 + \left(\frac{s_n}{2}\right)^2 = 1 \implies x = 1 - \sqrt{1 - \frac{s_n^2}{4}}$$

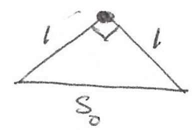
and plugging this into

$$\sqrt{x^2 + \left(\frac{s_n}{2}\right)^2} = s_{n+1} \implies s_{n+1} = \sqrt{2 - 2\sqrt{1 - s_n^2/4}}$$

(2) Let's now try it with a square:



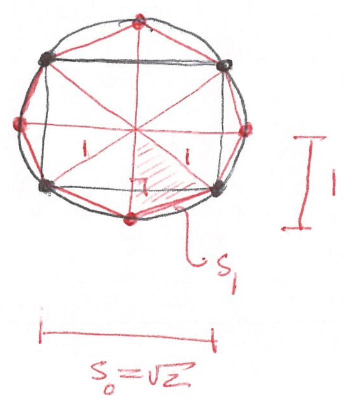

$\Rightarrow$



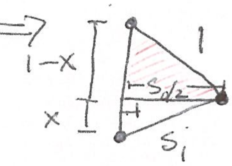

$$s_0^2 = 1^2 + 1^2 = 2$$

$$\Rightarrow \quad s_0 = \sqrt{2}$$

Now double the number of sides:




$\Rightarrow$




$$(1-x)^2 + (s_0/2)^2 = 1^2$$

$$\Rightarrow \quad \cancel{1} - 2x + x^2 + \frac{s_0^2}{4} = \cancel{1}$$

only – makes sense

$$\Rightarrow \quad x = \frac{2 \pm \sqrt{4 - s_0^2}}{2}$$

$$= 1 - \sqrt{1 - s_0^2/4}$$

& secondly

$$x^2 + (s_0/2)^2 = s_1^2 \quad \Rightarrow \quad s_1 = \sqrt{2 - 2\sqrt{1 - s_0^2/4}}$$

as with the hexagon. Similarly,

$$\boxed{s_{n+1} = \sqrt{2 - 2\sqrt{1 - s_n^2/4}}}$$

Therefore, in the first case

$$\boxed{\begin{aligned} s_0 &= 1 \\ s_{n+1} &= \sqrt{2 - 2\sqrt{1 - s_n^2/4}} \end{aligned}}$$

& in the second case

$$\boxed{\begin{aligned} s_0 &= \sqrt{2} \\ s_{n+1} &= \sqrt{2 - 2\sqrt{1 - s_n^2/4}} \end{aligned}}$$

So in both cases,

Sol'ns for (1) & (2)

$$\boxed{g(x) = \sqrt{2 - 2\sqrt{1 - x^2/4}}}$$

and consequently

$$\boxed{\text{Case 1} \begin{cases} f(0) = 1 \\ f(s(n)) = g(f(n)) \end{cases} \qquad \text{Case 2} \begin{cases} f(0) = \sqrt{2} \\ f(s(n)) = g(f(n)) \end{cases}}$$

(3) Proofs have been typed up & posted in the updated version of 'Numbers'

Sec. 11.1

(2) $A = \{1, 2, 3, 4, 5, 6\}$, $R =$ "divides",

e.g. $1 R 2$ bec. $1 | 2$ bec. $1 = 1 \cdot 2$.

Thus, since 1 divides every element of $A$, 2 divides 4 & 6, 3 divides 6, & every $a \in A$ divides itself,

$$\boxed{\begin{aligned} R = \{ &(1,1),\ (1,2),\ (1,3),\ (1,4),\ (1,5),\ (1,6), \\ &(2,2),\ (2,4),\ (2,6),\ (3,3),\ (3,6), \\ &(4,4),\ (5,5),\ (6,6) \} \end{aligned}}$$

(10) $R = \mathbb{R}^2 - \Delta$ where $\Delta = \{(x,x) \mid x \in \mathbb{R}\}$

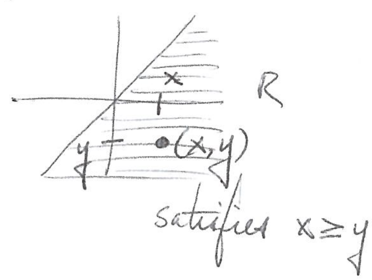
$\Longrightarrow$ $\boxed{x R y \text{ iff } x \neq y}$

Thus,

$$\boxed{R = \text{"}\neq\text{"}}$$
on $\mathbb{R}$

(12) $(x,y) \in R \iff y \leq x$ (or $x \geq y$). Thus,

$$\boxed{R = \text{"}\geq\text{"}}$$
on $\mathbb{R}$




(14) $(n,m) \in R \iff n \leq m$

$\iff$ $\boxed{\begin{array}{c} R = \text{"}\leq\text{"} \\ \text{on } \mathbb{Z} \end{array}}$

## <u>Sec. 11.2</u>

(2) $R$ is <u>not reflexive</u> since $(a,a) \notin R$,

is <u>not symmetric</u> since $(a,b) \in R$ but $(b,a) \notin R$

<u>is</u> transitive.

(8) $R$ on $\mathbb{Z}$ : $(x,y) \in R \overset{\text{def}}{\iff} |x-y| < 1$

$\iff x = y$

$\iff \boxed{R = \text{"="}}$

since $x \neq y$ in $\mathbb{Z}$ means $|x-y| \geq 1$.

(12) Let $R = \text{"|"}$ (divides, e.g. $2|4$) on $\mathbb{Z}$.

The def. of "|" is

$$\boxed{n|m \overset{\text{def}}{\iff} m = kn \text{ for some } k \in \mathbb{Z}}$$

Now,

$n|n$ bec. $n = 1 \cdot n$ for any $n \in \mathbb{Z}$

so "|" is reflexive. It is also transitive for

$n|m$ & $m|p \Rightarrow m = kn$, $p = lm$ for some $k, l \in \mathbb{Z}$

$\Rightarrow p = lm = l(kn) = (lk)n$ & $lk \in \mathbb{Z}$

$n|p. \Longleftarrow$

Sec. 11.3

(8) $R$ on $\mathbb{Z}$: $xRy \overset{def}{\Longleftrightarrow} x^2+y^2=2k$ for some $k \in \mathbb{Z}$

(a) $x^2+x^2=2x^2 \Longrightarrow xRx$

(b) $xRy \Longrightarrow x^2+y^2=2k$

$\Longrightarrow y^2+x^2=x^2+y^2=2k$

$\Longrightarrow yRx$

(c) $xRy$ & $yRz \Longrightarrow x^2+y^2=2k$

$y^2+z^2=2l$

for some $k, l \in \mathbb{Z}$

$$\Longrightarrow x^2+z^2=(x^2+y^2)+(y^2+z^2)-2y^2$$

$$=2k+2l-2y^2$$

$$=2\underbrace{(k+l-y^2)}_{\in \mathbb{Z}}$$

(10) Let $R$ & $S$ be equivalence relations on a set $X$, & consider $R \cap S$:

(a) $xRx$ & $xSx$ since both $R$ & $S$ are reflexive $\Longrightarrow (x,x) \in R$ & $(x,x) \in S$

$\Longrightarrow (x,x) \in R \cap S$

(b) $(x,y) \in R \Longrightarrow (y,x) \in R$ since $R$ & $S$

& $(x,y) \in S \Longrightarrow (y,x) \in S$ are symmetric

Therefore, $(x,y) \in R \cap S$

$\Longrightarrow (y,x) \in R \cap S$

(c) $(x,y), (y,z) \in R \cap S \Longrightarrow (x,y), (y,z) \in R$

& $(x,y), (y,z) \in S$

$\Longrightarrow (x,z) \in R$ & $(x,z) \in S$

bec. $R$ & $S$ are transitive

$\Longrightarrow (x,z) \in R \cap S$.

(12) If $R$ & $S$ are equivalence relations on $X$, then $R \cup S$ is in general not an equivalence relation.

Certainly $R \cup S$ is reflexive, since $(x,x) \in R \cap S$
$\subseteq R \cup S$
for any $x \in X$.

Certainly $(x,y) \in R \cup S \Rightarrow (x,y) \in R$ or $(x,y) \in S$
$\Rightarrow (y,x)\ R$ or $(y,x) \in S$
$\Rightarrow (y,x) \in R \cup S$

However, <u>transitivity</u> can fail: let

$$(x,y) \in R \subseteq \mathbb{R}^2 \overset{\text{def}}{\Longleftrightarrow} x^2 = y^2$$

$$(x,y) \in S \subseteq \mathbb{R}^2 \overset{\text{def}}{\Longleftrightarrow} x - y = 3k$$
for some $k \in \mathbb{Z}$

Check that these are equivalence relations on $\mathbb{R}$ but $(-1,1) \in R \subseteq R \cup S$, $(1,4) \in S \subseteq R \cup S$
yet $(-1,4) \notin R \cup S$.