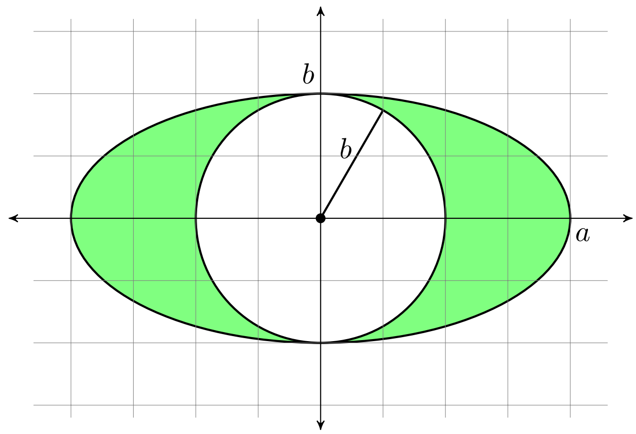
MATH 2300 - Homework 4

Instructions: This homework is due on Friday, June 29th. You may work with other students, however each person is responsible for writing their own solutions. Please write the names of any students who helped you.

1. Consider the circle $x^2 + y^2 = b^2$ sitting inside the ellipse $\dfrac{x^2}{a^2} + \dfrac{y^2}{b^2} = 1$,



   (a) Find the area between the circle and the ellipse (in green) by using the methods of section 7.4 (see Example 8, p. 356-357): in the first quadrant, the area is given by the integral of the difference of the functions $y = \sqrt{b^2 - \frac{b^2x^2}{a^2}}$ and $y = \sqrt{b^2 - x^2}$, so just multiply this result by 4 to get the total area.

   (b) Find the area between the circle and the ellipse (in green) by using the method of (horizontal) slicing (section 8.1). You may work in either top or bottom half and multiply the result by 2 if you wish, or in the first quadrant and multiply the result by 4.

   (c) Find the area between the circle and the ellipse (in green) by using polar coordinates (section 8.3). [Hint: Work in the first quadrant and multiply the result by 4. Plug $x = r\cos\theta$ and $y = r\sin\theta$ into the equation $\frac{x^2}{a^2} + \frac{y^2}{b^2} = 1$ and solve for r to get r as a function of θ, $r = f(\theta)$. You may use the fact that $\int \frac{1}{1+\alpha\sin\theta}\,d\theta = \frac{\arctan(\sqrt{\alpha+1}\tan\theta)}{\sqrt{\alpha+1}} + C$. Note: When evaluating this between 0 and $\pi/2$, you have to evaluate it between 0 and c, taking the limit as $c \to \pi/2$, because $\tan(\pi/2)$ is not defined.]

2. Consider again the ellipse $\dfrac{x^2}{a^2} + \dfrac{y^2}{b^2} = 1$.

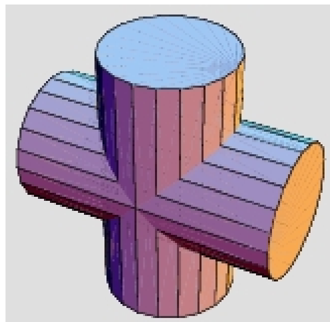
   (a) What are the cartesian coordinates of the point on the ellipse where $\theta = \pi/3$? [Hint: Look at the hint in 1(c) for the formula for r, then use the fact that $x = r\cos\theta$, $y = r\sin\theta$.]

   (b) Use implicit differentiation to find the derivative dy/dx at the point on the ellipse where $\theta = \pi/3$.

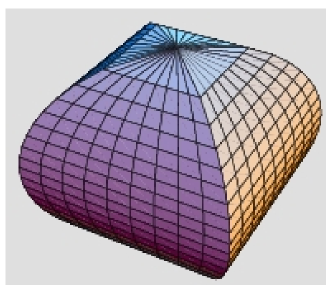
   (c) Now use the fact that $\dfrac{dy}{dx} = \dfrac{dy/d\theta}{dx/d\theta}$ to find the derivative at the point on the ellipse where $\theta = \pi/3$. Your result should match that of part (b).

3. Find the circumference, to two decimal places, of the ellipse $\dfrac{x^2}{16} + \dfrac{y^2}{9} = 1$. [Hint: Use the parametric equations $x = 4\cos t$, $y = 3\sin t$, and the fact that arc length (i.e. distance traveled) $= \displaystyle\int_0^{2\pi} v(t)\,dt = \int_0^{2\pi} \sqrt{\left(\frac{dx}{dt}\right)^2 + \left(\frac{dy}{dt}\right)^2}\,dt$. This integral will have to be evaluated numerically. You may use your calculator or a computer to do this.]

4. Two perpendicular cylinders of the same radius r, $x^2 + y^2 = r^2$ and $y^2 + z^2 = r^2$, intersect near the origin,



The solid formed by their intersection is called the **Steinmetz solid**,



As Wikipedia tells us, Archimedes and Zu Chongzhi computed its volume independently without using calculus. Here is a link to a short article about Archimedes' method. Your task is to compute the volume of the Steinmetz solid using calculus. [Hint: Use the slicing method.]

5. Find the volume of the solid obtained by rotating the portion of the region bounded by $y = \sqrt[3]{x}$ and $y = \dfrac{x}{4}$ lying in the first quadrant about the y-axis.

6. Find both a formula (in terms of n) as well as a recursive definition of the sequence $1 - \frac{1}{3}$, $1 - \frac{1}{5}$, $1 - \frac{1}{9}$, $1 - \frac{1}{17}$, $1 - \frac{1}{33}$, ... Does this sequence converge? If yes, show that it is monotone and bounded, so that Theorem 9.1 in our textbook applies, then compute the limit. If not, explain why not.

7. An **infinite surd** is a recursively defined sequence,

$$s_1 = \sqrt{k + \sqrt{k}},$$
$$s_n = \sqrt{k + s_{n-1}}, \quad \text{for } n > 1$$

where k is a positive integer greater than 1. If we can show that the sequence is increasing and bounded, then we know, from Theorem 9.1 in our textbook, that the sequence converges. Let us demonstrate that these two conditions are satisfied and then let us find the limit of the sequence.

(a) Show that the sequence s_n is increasing as follows: Show first that $s_1 < s_2$. Next, assuming $s_{n-1} < s_n$, show that $s_n < s_{n+1}$. (This shows that $s_2 < s_3$, because $s_1 < s_2$, and therefore $s_3 < s_4$, and therefore $s_4 < s_5$ and so on for all n.) [Hint: Use the fact that $s_n = \sqrt{k + s_{n-1}}$ and $s_{n+1} = \sqrt{k + s_n}$, and look at $s_{n+1}^2 - s_n^2$.]

(b) Clearly the sequence s_n is bounded below by 0. Show that it is bounded above by k as follows: First show that $s_1 < k$. Next, assuming $s_{n-1} < k$ for some fixed $n > 1$, show that $s_n < k$ (this shows that $s_2 < k$ because $s_1 < k$, and therefore $s_3 < k$ because $s_2 < k$, and so on, so that $s_n < k$ for all n).

(c) Thus, we know that the sequence converges. Use this knowledge to find the limit of the sequence. [Hint: $\lim_{n\to\infty} s_n = \lim_{n\to\infty} s_{n-1}$, so give that limit a name, s, and find s using the fact that $s_n = \sqrt{k + s_{n-1}}$.]

(d) What is the limit of the surd having $s_1 = 5$?

(e) There is a weaker version of Theorem 9.1 in the book, which replaces the notion monotone as strictly increasing ($s_n > s_{n-1}$) or strictly decreasing ($s_n < s_{n-1}$) with the slightly weaker notion of monotone as nondecreasing ($s_n \geq s_{n-1}$) or nonincreasing ($s_n \leq s_{n-1}$). If you were to run through parts (a) and (b) above with this theorem in mind, you would be able to include $k = 1$ in the general statement about convergence as well. Supposing this, use your result from part (c) to show that the **golden ratio**, $\frac{1+\sqrt{5}}{2}$, is in fact the limit of the infinite surd with $s_1 = 1$, i.e.

$$\frac{1+\sqrt{5}}{2} = \sqrt{1 + \sqrt{1 + \sqrt{1 + \sqrt{1 + \cdots}}}}$$