HW 10 Evens

Sec. 12.1

(8) $f \overset{\text{def}}{=} \{(x,y) \in \mathbb{Z}^2 \mid x+3y=4\}$ is

fr. $\mathbb{Z}$ to $\mathbb{Z}$

a function, because to each $x \in \mathrm{Dom}(f)$, there is exactly one y,

$$x+3y=4 \iff \boxed{y = \frac{-1}{3}x+\frac{4}{3}}$$

Indeed, say y' were another potential pair of x, then

$$y' = -\frac{1}{3}x+\frac{4}{3} = y$$

$$\Rightarrow y' = y$$

(10) $f = \{(x^3, x) \mid x \in \mathbb{R}\}$ is a function fr $\mathbb{R}$ to $\mathbb{R}$, because to each $x^3 \in \mathrm{Dom}(f)$, there is exactly on $x \in \mathrm{Ran}(f)$,

$$x = (x^3)^{1/3}$$

for if $x' = f(x) = x$, then $x' = (x^3)^{1/3} = x$, and indeed,

$$f(x^3) = x \iff f(x) = x^{1/3}$$

Sec. 12.2

(8) $f: \mathbb{Z}^2 \to \mathbb{Z}^2$, $\quad f(m,n) = (m+n, 2m+n)$

is injective: $\quad f(m,n) = f(m',n') \Longrightarrow$

$$(m+n, 2m+n) = (m'+n', 2m'+n')$$

$$\Longrightarrow \left.\begin{aligned} m+n &= m'+m' \\ 2m+n &= 2m'+n' \end{aligned}\right\} \text{a system of 2 eq'n in 2 var's.}$$

$$\Longrightarrow \quad \begin{aligned} 2m+n &= 2m'+n' && \mathrm{I} \\ m+n &= m'+n' && \mathrm{II} \quad \mathrm{I}-\mathrm{II} \\ \hline m+0 &= m'+0 \end{aligned}$$

$$\Longrightarrow \quad \boxed{m = m'}$$

and

$$\begin{aligned} m+n &= m'+n' && \mathrm{I} \\ 2m+n &= 2m'+n' && \mathrm{II} \quad -2\cdot\mathrm{I}+\mathrm{II} \\ \hline 0+(-n) &= 0+(-n') \end{aligned}$$

$$\Longrightarrow \quad \boxed{n = n'}$$

$$\Longrightarrow \quad \boxed{(m,n) = (m',n')}$$

Next, we show f is surjective: let $(a,b) \in \mathbb{Z}^2$ and let us find $(m,n) \in \mathbb{Z}^2$ s.t. $f(m,n) = (a,b)$.

Thus, we need to solve

$$\boxed{(m+n,\ 2m+n) = (a,b)}$$

This means solving the system

$$\begin{aligned} m+n &= a \qquad \mathrm{I} \\ 2m+n &= b \qquad \mathrm{II} \end{aligned}$$

We can use row-reduction:

$-2\mathrm{I}+\mathrm{II} \Rightarrow \quad -n = b-2a$

or $\boxed{n = 2a-b \in \mathbb{Z}}$

while

$\mathrm{II}-\mathrm{I} \Rightarrow \quad \boxed{m = b-a \in \mathbb{Z}}$

Verify:

$$\begin{aligned} f(m,n) &= f(b-a, 2a-b) \\ &= (m+n, \ 2m+n) \\ &= ((b-a)+(2a-b), \ 2(b-a)+(2a-b)) \\ &= (a,b) \quad \checkmark \end{aligned}$$

(10) $f : \mathbb{R} - \{1\} \longrightarrow \mathbb{R} - \{1\}$,

$$f(x) = \left(\frac{x+1}{x-1}\right)^3$$

is <u>bijective</u>:

<u>pf:</u> (1) <u>f is injective</u>:

$$f(x_1) = f(x_2)$$

$$\Longrightarrow \left(\frac{x_1+1}{x_1-1}\right)^3 = \left(\frac{x_2+1}{x_2-1}\right)^3$$

$$\Longrightarrow \frac{x_1+1}{x_1-1} = \frac{x_2+1}{x_2-1}$$

$$\Longrightarrow (x_1+1)(x_2-1) = (x_2+1)(x_1-1)$$

$$\Longrightarrow \cancel{x_1x_2} + x_2 - x_1 \cancel{-1} = \cancel{x_1x_2} + x_1 - x_2 \cancel{-1}$$

$$\Longrightarrow x_2 - x_1 = x_1 - x_2 = -(x_2 - x_1)$$

$$\Longrightarrow x_2 - x_1 = 0$$

$$\Longrightarrow x_2 = x_1$$

(2) <u>f is surjective</u>: Let $y \in \mathbb{R} - \{1\}$

let us find $x \in \mathbb{R} - \{1\}$ s.t. $f(x) = y$:

$$y = f(x) = \left(\frac{x+1}{x-1}\right)^3$$

$$\Longrightarrow y^{1/3} = \left(\frac{x+1}{x-1}\right)$$

$$\Longrightarrow y^{1/3}(x-1) = x+1$$

$$\Longrightarrow x(y^{1/3}-1) = 1 + y^{1/3}$$

$$\Longrightarrow \boxed{x = \frac{1+y^{1/3}}{y^{1/3}-1}}$$

and since $x \neq 1$
$f(x) = y \neq 1$ either
(i) if $x > 1$, $y > 1$
or
(ii) if ~~[illegible]~~
or $-1 < x < 1$, $y < 0$
(iii) $x < -1$, $y < 1$

(i) sim., if $y > 1$, $x > 1$, if
ii) $-1 < y < 1$, $x < 0$
iii) if $y < -1$, $x > 1$

so $x \neq 1$.

Verify:

$$f(x) = \left(\frac{x+1}{x-1}\right)^3$$

$$= \left(\frac{\left(\frac{y^{1/3}+1}{y^{1/3}-1}\right)+1}{\left(\frac{y^{1/3}+1}{y^{1/3}-1}\right)-1}\right)^3 = \left(\frac{\frac{y^{1/3}+\cancel{1}+y^{1/3}-\cancel{1}}{\cancel{y^{1/3}-1}}}{\frac{\cancel{y^{1/3}}+1-\cancel{y^{1/3}}+1}{\cancel{y^{1/3}-1}}}\right)^3 =$$

$$= \left(\frac{\cancel{2}y^{1/3}}{\cancel{2}}\right)^3$$

$$= y$$

(12) $\theta : \{0,1\} \times \mathbb{N} \longrightarrow \mathbb{Z}$

$$\theta(a,b) = a - 2ab + b$$

(i) θ is not injective, for $(0,0) \neq (1,1)$ in $\{0,1\} \times \mathbb{N}$, but

$$\begin{aligned} \theta(0,0) &= 0 - 2\cdot 0 \cdot 0 + 0 \\ &= 0 \\ &= 1 - 2\cdot 1\cdot 1 + 1 \\ &= \theta(1,1) \end{aligned}$$

(ii) θ is surjective: since $\theta(0,b) = 0 - 2\cdot 0b + b = b \in \mathbb{N}$

we see that θ hits all positive integers, while since $\theta(1,b) = 1 - 2\cdot 1\cdot b + b = 1 - b$, θ hits all negatives, too. Formally, consider cases: let $k \in \mathbb{Z}$,

<u>Case 1</u>: $k > 0 \implies \exists (0,k) \in \{0,1\} \times \mathbb{N}$

s.t. $\theta(0,k) = k$. ($k \geq 1$ here)

<u>Case 2</u>: $k = 0 \implies \exists (0,0), (1,1) \in \{0,1\} \times \mathbb{N}$

s.t. $\theta(0,0) = \theta(1,1) = 0$.

$-\underbrace{k}_{} \geq 1 \implies -k+1 \geq 2$, so $\in \mathbb{N}$

<u>Case 3</u>: $k < 0 \implies \exists (1, -k+1) \in \{0,1\} \times \mathbb{N}$

~~s.t. $\theta(1,-k+1) = 1 - 2(1)((-k+1)) + (-k)+1$~~

$\cancel{= 1 - 2k - 2 + k + 1}$

$\cancel{= -k}$

$$\theta(1, (-k)+1) = 1 - 2 \cdot 1 \cdot ((-k)+1) + (-k)+1$$

$$= \cancel{1} + 2k - \cancel{2} - k + \cancel{1}$$

$$= k$$

(e.g. $\theta(1,2) = 1 - 2 \cdot 1 \cdot 2 + 2 = -1$

$\theta(1,3) = 1 - 2 \cdot 1 \cdot 3 + 3 = -2$

etc.)

(18) $f: \mathbb{N} \to \mathbb{Z}$ is bijective,

$$f(n) = \frac{(-1)^n(2n-1)+1}{4}$$

pf: First, note that f actually lands in $\mathbb{Z}$ (not obvious): $n \geq 1$ here, so

$$f(1) = \frac{(-1)^1(2\cdot 1-1)+1}{4}$$

$$= \frac{-1+1}{4} = 0 \in \mathbb{Z}$$

$$f(2) = \frac{(-1)^2(2\cdot 2-1)+1}{4}$$

$$= \frac{4}{4} = 1 \in \mathbb{Z}$$

and generally

case 1 $n = 2k$ $(k \in \mathbb{N})$ $\Rightarrow$ $f(n) = \dfrac{(-1)^{2k}(2(2k)-1)+1}{4}$

$$= \frac{4k-1+1}{4} = k \in \mathbb{Z}$$

case 2 $n = 2k+1$ $(k \in \mathbb{N})$ $\Rightarrow$ $f(n) = \dfrac{(-1)^{2k+1}(2(2k+1)-1)+1}{4}$

$$= \frac{-(4k+1)+1}{4} = -k \in \mathbb{Z}$$

Matter of fact, this preliminary analysis provides the proof of surjectivity:

(1) <u>f is surjective</u>: Let $k \in \mathbb{Z}$, & let us find $n \in \mathbb{N}$ s.t. $f(n) = k$.

<u>Case 1</u>: $k = 0 \Longrightarrow f(1) = 0$ $\quad (n=1)$

<u>Case 2</u>: $k > 0 \Longrightarrow f(2k) = k$ $\quad (n=2k)$

& $n = 2k \in \mathbb{N}$ since $k \in \mathbb{N}$

<u>Case 3</u>: $k < 0 \Longrightarrow f(2k+1) = -k$

$(n = 2k+1)$

(2) <u>f is injective</u>:

<u>Case 1</u>: $f(n) = f(m) = 0 \Longrightarrow$

$$\frac{(-1)^n(2n-1)+1}{4} = \frac{(-1)^m(2m-1)+1}{4}$$

$$\Longrightarrow (-1)^n(2n-1) = (-1)^m(2m-1)$$

$$\Longrightarrow (-1)^{n+m}(2n-1) = 2m-1$$

$$\Longrightarrow 2\left((-1)^{n+m}n - m\right) = -1 + (-1)^{n+m}$$

<u>Case 1</u> $n+m$ is even $\Longrightarrow$

$$2(n-m) = 0 \Longrightarrow$$

$$n-m = 0 \Longrightarrow$$

$$n = m$$

<u>Case 2</u> $n+m$ is odd $\Longrightarrow$

$$2(-n-m) = -2 \Longrightarrow$$

$$-n-m = -1 \Longrightarrow$$

$$n+m = 1$$

which contradicts the assumption that $n, m \in \mathbb{N}$, i.e. $n, m \geq 1$, so $n+m > 1$ and $n+m=1$ is a contradiction.

Thus, $n+m$ is even & therefore $n=m$.

Sec. 12.4

(6) $f, g : \mathbb{R} \to \mathbb{R}$

$$f(x) = \frac{1}{1+x^2}, \quad g(x) = 3x+2$$

$$\Longrightarrow \boxed{\begin{aligned}(f\circ g)(x) &= f(g(x)) \\ &= f(3x+2) \\ &= \frac{1}{1+(3x+2)^2} \\ &= \frac{1}{9x^2+12x+5}\end{aligned}}$$

&

$$\boxed{\begin{aligned}(g\circ f)(x) &= g(f(x)) \\ &= g\left(\frac{1}{1+x^2}\right) \\ &= 3\left(\frac{1}{1+x^2}\right) + 2 \\ &\quad \text{or} \\ &\quad \frac{2x^2+5}{x^2+1}\end{aligned}}$$

Sec. 12.4

(8) $f, g : \mathbb{Z}^2 \to \mathbb{Z}^2$,

$$f(m,n) = (3m - 4n,\ 2m + n)$$

$$g(m,n) = (5m + n,\ m)$$

$\Longrightarrow$ $f \circ g,\ g \circ f : \mathbb{Z}^2 \to \mathbb{Z}^2$

are given by

$$\boxed{\begin{aligned}(f \circ g)(m,n) &= f(g(m,n)) \\ &= f(5m+n,\ m) \\ &= \big(3(5m+n) - 4m,\ 2(5m+n) + m\big) \\ &= (11m + 3n,\ 11m + 2n)\end{aligned}}$$

and

$$\boxed{\begin{aligned}(g \circ f)(m,n) &= g(3m - 4n,\ 2m + n) \\ &= \big(5(3m-4n) + (2m+n),\ 2m+n\big) \\ &= (17m - 19n,\ 2m + n)\end{aligned}}$$

(10) $f: \mathbb{R}^2 \to \mathbb{R}^2$, $\quad f(x,y) = (xy, x^3)$

$$\Longrightarrow \quad (f \circ f)(x,y) = f(f(x,y))$$

$$= f(\underbrace{xy}_{=x'}, \underbrace{x^3}_{=y'})$$

$$= f(x', y')$$

$$= (x'y', (x')^3)$$

$$= ((xy) \cdot x^3, (xy)^3)$$

$$= \boxed{(x^4 y, x^3 y^3)}$$

Sec. 12.5

(2) $y = \dfrac{5x+1}{x-2} \iff y(x-2) = 5x+1$

$xy - 2y =$

$\iff x(y-5) = 1+2y$

$\iff x = \dfrac{1+2y}{y-5}$

$$\Longrightarrow \boxed{f^{-1}(x) = \frac{1+2x}{x-5}}$$

Verify: $(f \circ f^{-1})(x) = f\left(\dfrac{1+2x}{x-5}\right)$

$$= \frac{5\left(\frac{1+2x}{x-5}\right) + 1 \cdot \frac{x-5}{x-5}}{\left(\frac{1+2x}{x-5}\right) - 2 \cdot \frac{x-5}{x-5}}$$

$$= \frac{\left(\frac{\cancel{5}+10x+x-\cancel{5}}{\cancel{x-5}}\right)}{\left(\frac{1+\cancel{2x}-\cancel{2x}+10}{\cancel{x-5}}\right)}$$

$$= \frac{11x}{11} = x \quad \checkmark$$

and

$$(f^{-1}\circ f)(x) = f^{-1}(f(x))$$

$$= f^{-1}\left(\frac{5x+1}{x-2}\right)$$

$$\frac{x-2}{x-2}\cdot\;=\; \frac{1+2\left(\frac{5x+1}{x-2}\right)}{\left(\frac{5x+1}{x-2}\right)-5}\cdot\frac{x-2}{x-2}$$

$$= \frac{\dfrac{x\cancel{-2}+10x\cancel{+2}}{\cancel{x-2}}}{\dfrac{\cancel{5x}+1-\cancel{5x}+10}{\cancel{x-2}}}$$

$$= \frac{11x}{11}$$

$$= x \quad \checkmark$$

(4) $y = e^{x^3+1} \iff \ln y = x^3+1$

$\iff \ln y - 1 = x^3$

$\iff (\ln y - 1)^{1/3} = x$

so $\boxed{f^{-1}(x) = (\ln x - 1)^{1/3}}$

Verify: $(f \circ f^{-1})(x) = f(f^{-1}(x))$

$= f((\ln x - 1)^{1/3})$

$= e^{[(\ln x - 1)^{1/3}]^3+1}$

$= e^{\ln x - 1 + 1}$

$= e^{\ln x}$

$= x$ ✓

and $(f^{-1} \circ f)(x) = f^{-1}(f(x)) = f^{-1}(e^{x^3+1})$

$= [\ln(e^{x^3+1}) - 1]^{1/3}$

$= (x^3+1-1)^{1/3}$

$= x$ ✓

(6) $f : \mathbb{Z}^2 \to \mathbb{Z}^2$,

$$f(m,n) = (5m+4n,\ 4m+3n) = (a,b)$$

$$\Longleftrightarrow \quad \begin{aligned} 5m+4n &= a \qquad \mathrm{I} \\ 4m+3n &= b \qquad \mathrm{II} \end{aligned}$$

Now, row-reduce: $-\frac{4}{5}\mathrm{I} + \mathrm{II} =$

$$\begin{aligned} -4m - \tfrac{16}{5}n &= -\tfrac{4a}{5} \\ + \quad 4m + 3n &= b \end{aligned}$$

$$\left(\frac{-16}{5} + 3\cdot\frac{5}{5}\right)n = -\frac{4}{5}a + b$$

$$\frac{-16+15}{5}n =$$

$$\frac{-1}{5}n =$$

mult. by 5

$$\Longrightarrow \quad -n = -4a + 5b$$

$$\Longrightarrow \quad \boxed{n = 4a - 5b}$$

Plug back into, e.g. $4m+3n=b$ to get m:

$$4m + 3(4a-5b) = 4m+3n = b$$

$$\Longrightarrow \quad \boxed{m = \frac{b - 3(4a-5b)}{4} = \frac{-12a+16b}{4} = 4b - 3a}$$

$$\Rightarrow \quad (m,n) = (4b-3a,\ 4a-5b)$$

i.e. $$\boxed{f^{-1}(m,n) = (4n-3m,\ 4m-5n)}$$

<u>Verify:</u>

$$(f \circ f^{-1})(m,n) = f(\underbrace{4n-3m}_{=m'},\ \underbrace{4m-5n}_{=n'})$$

$$= (5m'+4n',\ 4m'+3n')$$

$$= (5(4n-3m)+4(4m-5n),$$
$$4(4n-3m)+3(4m-5n))$$

$$= (\cancel{20n}\underbrace{-15m+16m}_{=m}-\cancel{20n},$$
$$\underbrace{\cancel{16n}-\cancel{12m}+\cancel{12m}-\cancel{15n}}_{=n})$$

$$= (m,n) \checkmark$$

and

$$(f^{-1} \circ f)(m,n) = f^{-1}(f(m,n))$$

$$= f^{-1}(\underbrace{5m+4n}_{=m'}, \underbrace{4m+3n}_{=n'})$$

$$= (4n' - 3m', 4m' - 5n')$$

$$= (4(4m+3n) - 3(5m+4n),$$
$$4(5m+4n) - 5(4m+3n))$$

$$= (16m + \cancel{12n} - 15m - \cancel{12n},$$
$$\cancel{20m} + 16n - \cancel{20m} - 15n)$$

$$= (m,n) \checkmark$$

Sec. 12.6

(6) $f: A \to B, \quad Y \subseteq B$

$\Rightarrow \boxed{f(f^{-1}(Y)) \subseteq Y}$ but not always $= Y$

pf: Let $y \in f(f^{-1}(Y))$

$= \{ z \in B \mid f(x) = z$ for some $x \in f^{-1}(Y) \}$

so that

$y = f(x)$ for $x \in f^{-1}(Y)$

$= \{ x \in A \mid f(x) \in Y \}$

$\Rightarrow f(x) = y \in Y$

counterexample

However, consider

$f: \underbrace{\mathbb{R}}_{=A} \to \underbrace{\mathbb{R}}_{=B}, \quad f(x) = x^2, \quad Y = [-1,1] \subseteq \underbrace{\mathbb{R}}_{=B}$

$\Rightarrow f^{-1}(Y) = \{ x \in \mathbb{R} \mid x^2 \in [-1,1] \}$

$= [-1,1]$

But

$$x \in [-1,1] = Y \implies f(x) = x^2 \in [0,1] \subsetneq [-1,1] = Y$$

$$\implies f(f^{-1}(Y)) = [0,1] \subsetneq [-1,1] = Y$$

(22) $f: A \to B,\ W, X \subseteq A \implies$

$$f(W \cap X) \subseteq f(W) \cap f(X)$$

but not always $=$:

pf: Let $y \in f(W \cap X) = \{z \in B \mid z = f(x) \text{ for some } x \in W \cap X\}$

$\implies y = f(x)$ & $x \in W$ & $x \in X$

$\implies y \in f(W)$ and $y \in f(X)$

$\implies y \in f(W) \cap f(X)$.

However, to see that we do not always have equality, consider the following counterexample:

$$f: \underset{=A}{\mathbb{R}} \to \underset{=B}{\mathbb{R}}, \quad f(x) = x^2$$

$$\underbrace{W = (-1, 0), \quad X = (0,1)}_{\text{intervals}} \subseteq A = \mathbb{R}$$

Since $W \cap X = \emptyset$, $f(W \cap X) = \emptyset$ too, whereas $f(W) = f(X) = (0,1)$

$$\Longrightarrow \quad f(W) \cap f(X) = (0,1) \neq \emptyset$$
$$= f(W \cap X).$$

(10) $f: A \to B, \quad Y, Z \subseteq B \implies$

$$f^{-1}(Y \cap Z) = f^{-1}(Y) \cap f^{-1}(Z)$$

pf: $f^{-1}(Y \cap Z) = \{x \in A \mid f(x) \in Y \cap Z\}$

$= \{x \in A \mid f(x) \in Y \ \& \ f(x) \in Z\}$

$= \{x \in A \mid f(x) \in Y\} \cap \{x \in A \mid f(x) \in Z\}$

$= f^{-1}(Y) \cap f^{-1}(Z)$

(12) $f: A \to B$ is injective iff $X = f^{-1}(f(X))$, $\forall X \subseteq A$

& surjective iff

$f(f^{-1}(Y)) = Y, \ \forall Y \subseteq B$

~~pf: (1) f is injective $\overset{\text{def}}{\iff}$ ($f(x_1) = f(x_2) \implies x_1 = x_2$, $\forall x_1, x_2 \in A$) $\implies \forall x \in f^{-1}(f(X))$ ($X \subseteq A$) we have $f(x) \in f(X)$ by the def. of $f(X) = \{y \in B \mid y = f(x), x \in X$~~

pf: (1) Suppose f injective & let $X \subseteq A$.

We must show $X = f^{-1}(f(X))$. By #5 above, we know that, always,

$$X \subseteq f^{-1}(f(X))$$

so it remains to show that when f is injective, $\boxed{f^{-1}(f(X)) \subseteq X}$, too. Let

pf of $f^{-1}(f(X)) \subseteq X$

$$x \in f^{-1}(f(X)) = \{a \in A \mid f(a) \in f(X)\}$$

$$= \{a \in A \mid f(a) = f(b) \text{ for some } b \in X\}$$

Then, by injectivity of f we conclude that

$$f(a) = f(b) \implies a = b \in X$$

& $x = a$ for one of these a's, so

$$x = a \in X.$$

Thus, we have shown that

$$\boxed{f \text{ injective} \implies X = f^{-1}(f(X)) \quad \forall X \in \mathcal{P}(A)}$$

Now for the converse: suppose $X = f^{-1}(f(X))$
$\forall X \in \mathcal{P}(A)$

and consider

$$f(x_1) = f(x_2)$$

for some $x_1, x_2 \in A$. We want to show that $x_1 = x_2$. Let

$$X \overset{\text{def}}{=} \{x_1\} \in \mathcal{P}(A)$$

Then, by our assumption, $X = f^{-1}(f(X))$.
However, $f(X) = \{f(x_1)\} = \{f(x_2)\}$, while

$$x_2 \in \{x_1\} = X = f^{-1}(f(X)) = \{x \in A \mid f(x) \in f(X)\}$$
$$= \{x \in A \mid f(x) = f(x_1)\}.$$

$\implies x_1 = x_2$

Thus,

$$\boxed{\begin{aligned} & X = f^{-1}(f(X)),\ \forall X \in \mathcal{P}(A) \\ & \Longrightarrow \left(f(x_1) = f(x_2) \Longrightarrow x_1 = x_2, \text{ applying to } X = \{x_1\} \right) \\ & \Longrightarrow f \text{ injective} \end{aligned}}$$

(2) Now suppose f is surjective, meaning $\forall b \in B,\ \exists a \in A$ s.t. $f(a) = b$, & let us show that $f(f^{-1}(Y)) = Y,\ \forall Y \in \mathcal{P}(B)$.
We already, in #6 above, that

$$f(f^{-1}(Y)) \subseteq Y, \quad \forall Y \in \mathcal{P}(B)$$

so it remains to show

$$Y \subseteq f(f^{-1}(Y))$$

Let $y \in Y$. By surjectivity, $\exists x \in A$ s.t. $f(x) = y$. But this shows that

$$x \in f^{-1}(Y)$$

So

$$y = f(x) \in f(f^{-1}(Y))$$

$$= \{ z \in B \mid z = f(a) \text{ for some } a \in f^{-1}(Y) \}$$

$\Longrightarrow$

$$Y \subseteq f(f^{-1}(Y)), \quad \forall Y \in \mathcal{P}(B).$$

Conversely, supp. $Y \subseteq f(f^{-1}(Y)), \ \forall Y \in \mathcal{P}(B)$ & let $y \in B$. Apply the assumption to $Y = \{y\} \in \mathcal{P}(B)$:

$$\{y\} = f(f^{-1}(\{y\}))$$
$$= \{ z \in B \mid z = f(x), \text{ for some } x \in f^{-1}(\{y\}) \}$$

$\Longrightarrow\ y = f(x)$ for some $x \in f^{-1}(\{y\}) \subseteq A$

$\Longrightarrow\ f$ is surjective.

(14) Let $f: A \to B$ & $Y \subseteq B$. Then,

$$\boxed{f^{-1}(f(f^{-1}(Y))) = f^{-1}(Y)}$$

pf: Observe first that

$$\begin{aligned} f^{-1}(f(f^{-1}(Y))) &= \{x \in A \mid f(x) \in \underbrace{f(f^{-1}(Y))}_{\subseteq Y \text{ by } (6)}\} \\ &\subseteq \{x \in A \mid f(x) \in Y\} \\ &= f^{-1}(Y) \end{aligned}$$

$$\Longrightarrow\ \boxed{f^{-1}(f(f^{-1}(Y))) \subseteq f^{-1}(Y)}$$

Now let $x \in f^{-1}(Y) = \{a \in A \mid f(a) \in Y\}$, so $f(x) \in Y$. Then,

$$f(x) \in f(f^{-1}(Y)) = \{z \in B \mid z = f(a) \text{ for some } a \in f^{-1}(Y)\}$$

$$\Longrightarrow \quad x \in f^{-1}(f(f^{-1}(\cancel{Y}))) = \{a \in A \mid f(a) \in f(f^{-1}(Y))\}$$

$$\Longrightarrow \quad \boxed{f^{-1}(Y) \subseteq f^{-1}(f(f^{-1}(Y)))}$$

& therefore

$$\boxed{f^{-1}(Y) = f^{-1}(f(f^{-1}(Y)))}$$

<u>Sec. 13.2</u>

(2) $\lim\limits_{x \to -1} (4x+6) = 2$

<u>pf:</u> Let $\varepsilon > 0$ be given, & let us find $\delta > 0$ s.t. $|x+1| = |x-(-1)| < \delta \Rightarrow |(4x+6) - 2| < \varepsilon$

Since $4x+6-2 = 4x+4 = 4(x+1)$,

to get $|4x+6-2| = 4|x+1|$ less than ε,

make

$$\delta \overset{\text{def}}{=} \frac{\varepsilon}{4}$$

Then,

$$|x+1| < \delta \implies 4|x+1| < 4\delta$$
$$= 4 \cdot \frac{\varepsilon}{4}$$
$$= \varepsilon$$

$$\implies |(4x+6)-2| < \varepsilon.$$

(6) $\lim_{x \to 1} (4x^2+1) = 5$

pf: Let $\varepsilon > 0$ be given, & observe

$$4x^2+1 = 4x^2 - 4 + 5$$
$$= 4(x^2-1) + 5$$
$$= 4(x-1)(x+1) + 5$$

$$\implies (4x^2+1) - 5 = 4(x-1)(x+1)$$
$$\implies |4x^2+1-5| = 4|x-1||x+1|.$$

i.e. $\lim_{x\to 1}(x+1)=2$

Since $x+1$ approaches 2 as x approaches 1,

(pf: $\forall \varepsilon>0, \exists \delta'=\varepsilon$ s.t. $|x-1|<\delta' \Longrightarrow$

$|(x+1)-2| = |x-1| < \delta' = \varepsilon'$)

if we choose $\varepsilon'=1$, then $\delta'=1$ &

$$\underbrace{|x-1|<1}_{=\delta'} \Longrightarrow |x+1| = |x-1+2|$$

$$\leq |x-1|+|2|$$

$$< 1+2$$

$$= 3$$

so if $|x-1|<1$, we have

$$|4x^2+1-5| = 4|x-1||x+1|$$

$$< 4|x-1|\cdot 3$$

$$= 12|x-1|$$

Therefore, for our original $\varepsilon>0$, let

$$\delta = \min\{1, \varepsilon/12\}$$

& observe

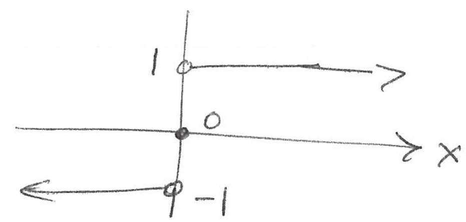
$$|4x^2+1-5| < 12|x-1| < 12\delta \leq \cancel{12}\cdot\frac{\varepsilon}{\cancel{12}}$$

$$= \varepsilon.$$

Sec. 13.3

(2) $\lim_{x \to 0} \frac{|x|}{x}$ DNE

First, observe that $\frac{|x|}{x} = \begin{cases} 1, & \text{if } x > 0 \\ -1, & \text{if } x < 0 \end{cases}$



If we suppose $\lim_{x \to 0} \frac{|x|}{x} = L \in \mathbb{R}$, then

$\forall \varepsilon > 0, \exists \delta > 0$ s.t.

$$0 < |x - 0| < \delta \implies \left| \frac{|x|}{x} - L \right| < \varepsilon$$

Well, try $\varepsilon = \frac{1}{2}$, say, & observe that

$$0 < |x| < \delta \implies |\pm 1 - L| < \tfrac{1}{2}$$

$|L - 1| < \frac{1}{2}$

&

$|L + 1| < \frac{1}{2}$

& $|1 + L| = |-1 - L| < \frac{1}{2}$

& $|L - 1| = |-L + 1| < \frac{1}{2}$

But this is impossible since

$$|L-1| < 1/2 \iff -1/2 < L-1 < 1/2$$

$$\iff 1-1/2 < L < 1+1/2$$

$$\iff 1/2 < L < 3/2$$

while

$$|L+1| < 1/2 \iff -1/2 < L+1 < 1/2$$

$$\iff -1-1/2 < L < 1/2-1$$

$$\iff -3/2 < L < -1/2$$

for then we would have

$$L < -1/2 < 1/2 < L$$

$$\Rightarrow L < L.$$

<u>Sec. 13-3</u>

(4) $\lim_{x\to\pi/2} \cos(\frac{1}{x})$ <u>does</u> exist, it equals $\cos(\frac{2}{\pi})\in\mathbb{R}$!

(Hammack typo)

<u>Sec. 13.4</u>

(2) Sketch: Apply Thm 13-7 for the base case $n=2$, then again at the inductive step on

$$\underbrace{(f_1(x)\cdots f_n(x))}_{f}\cdot \underbrace{f_{n+1}(x)}_{g}$$

(4) Apply Thms. 13.2–13.8, or alternatively #3 & Thm. 13-8:

$$\lim_{x\to c}\frac{f(x)}{g(x)} = \frac{\lim_{x\to c} f(x)}{\lim_{x\to c} g(x)} = \frac{f(c)}{g(c)}$$