Formula Sheet

1 The Three Defining Properties of Real Numbers

For all real numbers a, b and c, the following properties hold true.

1. The **commutative property**:

$$a + b = b + a$$
$$ab = ba$$

2. The **associative property**:

$$a + (b + c) = (a + b) + c$$
$$a(bc) = (ab)c$$

3. The **distributive property**:

$$a(b + c) = ab + ac$$

These properties define (almost) all other facts about real numbers, and chief among them are the formulas we give below.

Remember that the distributive property is an equality between two expressions. Going from the left expression to the right expression is called **distributing** a over the sum $b + c$, while going from the right expression to the left expression is called **factoring** a out, and a is called the **common factor** of ab and ac.

2 Important Formulas

Here are the **factoring formulas** you should know by now: for any real numbers a and b,

$(a + b)^2 = a^2 + 2ab + b^2$	Square of a Sum
$(a - b)^2 = a^2 - 2ab + b^2$	Square of a Difference
$a^2 - b^2 = (a - b)(a + b)$	Difference of Squares
$a^3 - b^3 = (a - b)(a^2 + ab + b^2)$	Difference of Cubes
$a^3 + b^3 = (a + b)(a^2 - ab + b^2)$	Sum of Cubes

And here are the **exponential rules** you should know: for any real numbers a and b, and any rational numbers $\frac{p}{q}$ and $\frac{r}{s}$,

$$\boxed{\begin{array}{rcll} a^{p/q}a^{r/s} & = & a^{p/q+r/s} & \text{Product Rule} \\ & = & a^{\frac{ps+qr}{qs}} & \\ \dfrac{a^{p/q}}{a^{r/s}} & = & a^{p/q-r/s} & \text{Quotient Rule} \\ & = & a^{\frac{ps-qr}{qs}} & \\ (a^{p/q})^{r/s} & = & a^{pr/qs} & \text{Power of a Power Rule} \\ (ab)^{p/q} & = & a^{p/q}b^{p/q} & \text{Power of a Product Rule} \\ \left(\dfrac{a}{b}\right)^{p/q} & = & \dfrac{a^{p/q}}{b^{p/q}} & \text{Power of a Quotient Rule} \\ a^0 & = & 1 & \text{Zero Exponent} \\ a^{-p/q} & = & \dfrac{1}{a^{p/q}} & \text{Negative Exponents} \\ \dfrac{1}{a^{-p/q}} & = & a^{p/q} & \text{Negative Exponents} \end{array}}$$

Remember, there are different notations:

$$\boxed{\begin{array}{l} \sqrt[q]{a} = a^{1/q} \\ \sqrt[q]{a^p} = a^{p/q} = (a^{1/q})^p \end{array}}$$

For example, the power of a product rule in radical notation would be

$$\sqrt[q]{(ab)^p} = \sqrt[q]{a^p}\sqrt[q]{b^p}$$

3 Quadratic Formula

Finally, the **quadratic formula**: if a, b and c are real numbers, then the quadratic polynomial equation

$$ax^2 + bx + c = 0$$

has (either one or two) solutions

$$\boxed{x = \frac{-b \pm \sqrt{b^2 - 4ac}}{2a}}$$

The **discriminant** is the number under the square root, $b^2 - 4ac$.

1. If $b^2 - 4ac > 0$, there are two real roots, possibly (indeed quite likely) irrational.
2. If $b^2 - 4ac = 0$, there is one real root, namely $-b/2a$, and it is rational if a and b are.
3. If $b^2 - 4ac < 0$, there are two complex roots, so no real roots.

4 Points and Lines

Suppose you have two points in the plane,

$$P = (x_1, y_1), \quad Q = (x_2, y_2)$$

What information can you get from them? Three things:

1. The **distance** between them, $d(P, Q) = \sqrt{(x_2 - x_1)^2 + (y_2 - y_1)^2}$.
2. The coordinates of the **midpoint** between them, $M = \left(\dfrac{x_1 + x_2}{2}, \dfrac{y_1 + y_2}{2}\right)$.
3. The **slope** of the line through them, $m = \dfrac{y_2 - y_1}{x_2 - x_1} = \dfrac{\text{rise}}{\text{run}}$.

This information comes in handy when doing line problems, especially the slope part. As for **lines**, remember, they can be represented in three different ways:

Standard Form	$ax + by = c$
Slope-Intercept Form	$y = mx + b$
Point-Slope Form	$y - y_1 = m(x - x_1)$

where a, b, c are real numbers, m is the slope, b (different from the standard form b) is the y-intercept, and (x_1, y_1) is *any* fixed point on the line. The first one is basically only useful for finding the x- and y-intercepts (by letting $y = 0$ and $x = 0$, respectively). The second is useful for graphing and finding the x- and y-intercepts (also by letting $y = 0$ and $x = 0$, respectively). The third is useful for coming up with the equation of a line given only information about a point and a slope, or equivalently, by (3), given information about two points.

Suppose two lines ℓ_1 and ℓ_2 have slope-intercept forms $y = m_1 x + b_1$ and $y = m_2 x + b_2$. Then ℓ_1 and ℓ_2 are **parallel**, denoted $\ell_1 \| \ell_2$, if their slopes are the same, that is if $m_1 = m_2$, and they **perpendicular**, denoted $\ell_1 \perp \ell_2$, if their slopes are negative reciprocals, that is if $m_1 = -\dfrac{1}{m_2}$ (or equivalently $m_2 = -\dfrac{1}{m_1}$).

Example 4.1 *Suppose I'm given two points*

$$P = (22, 12), \quad Q = (4, -15)$$

and asked to come up with the equation of the line passing through them and then graph it. In order to come up with the equation, I need the slope. I can't do anything without that. Luckily, I can get the slope using (3):

$$m = \frac{-15 - 12}{4 - 22} = \frac{-27}{-18} = \frac{3}{2}$$

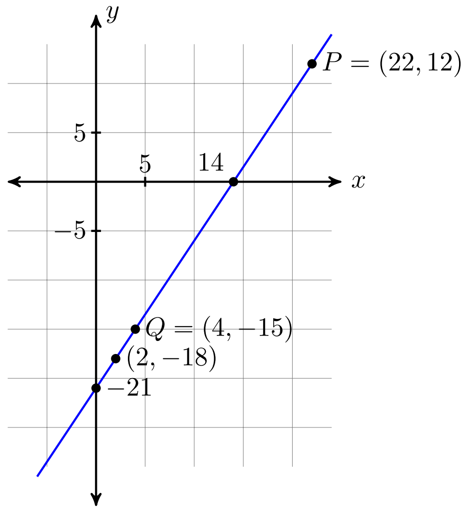
Now I've got a couple of points and I've got a slope, so I naturally use point-slope to give a rough-draft version of the equation (the final draft will be slope-intercept here, since that's what I need to graph it): picking $(22, 12)$ *for no particular reason, I get*

$$y - 12 = \frac{3}{2}(x - 22)$$

Simplifying gives

$$\boxed{y = \frac{3}{2}x - 21}$$

This I can graph: I go to $(0,-21)$, because my y-intercept is -21, which means $x = 0$ there, and plot a point there. Then I go up three and over two and plot a point there (i.e. at $(2,-18)$), then I connect the dots, and I'm done.



5 Circles

We know from Euclidean geometry that a **circle**, sometimes denoted $\odot$, is by definition the set of all points $X := (x, y)$ a fixed distance r, called the **radius**, from another given point $C = (h, k)$, called the **center** of the circle,

$$\bigodot \stackrel{\text{def}}{=} \{X \mid d(X, C) = r\} \tag{5.1}$$

Using the distance formula (1) and the square root property, $d(X,C) = r \iff d(X,C)^2 = r^2$, we see that this is precisely

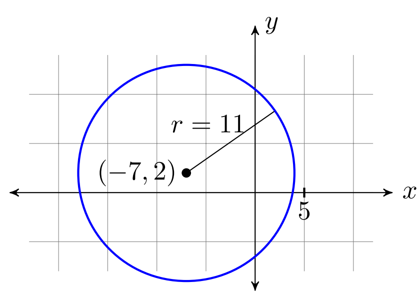
$$\bigodot \stackrel{\text{def}}{=} \{(x, y) \mid (x-h)^2 + (y-k)^2 = r^2\} \tag{5.2}$$

which gives the familiar equation for a circle.

Example 5.1 *Suppose $C = (-7, 2)$ and $r = 11$. The equation of this circle is*

$$\boxed{(x+7)^2 + (y-2)^2 = 121}$$

and it's graph is



6 Functions

An **ordered pair** (a, b) of numbers a and b (or other things, perhaps matrices or polynomials, or whatever) is like the set $\{a, b\}$ except that we're keeping tabs on the positions of a and b. One comes first, the other comes second.

A **cartesian product** $A \times B$ of two sets A and B is the set of all ordered pairs (a, b), i.e.

$$A \times B = \{(a, b) \mid a \in A,\ b \in B\}$$

A **relation** on $A \times B$ is *any* subset of $A \times B$. If $R = A' \times B'$ is a relation on $A \times B$, then the set A' is called the **domain**, and the set B' the **range** of the relation. I.e. the domain is the set of first coordinates, and the range is the set of second coordinates of R. The set B is called the **codomain** of the relation.

Example 6.1 *The set of points a distance greater than* 1 *from* $(0, 0)$ *is a relation on* $\mathbb{R} \times \mathbb{R}$*, since that's a subset of the plane. An ellipse in the plane is a relation on* $\mathbb{R} \times \mathbb{R}$*. Other examples are equality* $=$*, strict inequality* $<$ *and partial inequality* $\leq$ *on* $\mathbb{R} \times \mathbb{R}$*. For example, if we were to graph* $=$*, we'd get the line through the origin with slope of* 1*, i.e.* $y = x$*. Sometimes we have special notation for certain relations. For example, we usually write* $a = b$ *or* $a < b$ *or* $a \leq b$ *instead of* $(a, b) \in =$ *or* $(a, b) \in <$ *or* $(a, b) \in \leq$*, even though* $<$*, for example, really means a subset of the plane, i.e. set* $\{(a, b) \mid a$ *is strictly less than* $b\}$*.*

The most important relation by far is the function. A **function** $f : A \to B$ is actually a relation on $A \times B$, that is, it is a collection of ordered pairs (a, b) in $A \times B$. We employ the special function notation $f(a) = b$ instead of (a, b), to make sure we understand we're dealing with a function here and not just any relation. A function's ordered pairs $f(a) = b$ must satisfy a very important requirement: the "**vertical line test**", which is that if we run a vertical line across the graph of f, we can only hit one point at a time with it. In words, the vertical line test means there are no two points (a, b) and (a, c) in the relation f with $a \neq c$. Remember, an arbitrary relation f is just a bunch of ordered pairs, so it can happen that (a, b) and (a, c) with different b and c are in f. But then we just say f is not a function. Whatever else it is, it's not a function. This is in fact the essence of a function. We *want* functions to avoid having both $f(a) = b$ and $f(a) = c$ for $b \neq c$.

Given a real function f, i.e. a subset of $\mathbb{R} \times \mathbb{R}$, and given two ordered pairs (x_1, y_1) and (x_2, y_2) of f (i.e. $f(x_1) = y_1$ and $f(x_2) = y_2$), we define the **average rate of change** of f as x varies between x_1 and y_1 as the quotient

$$\frac{y_2 - y_1}{x_2 - x_1}$$

This is a lot like the slope between two points. In fact, it *is* the slope between the two points, except that in this context we pick the points to lie on the graph of f. The reason we call it *average* is we're ignoring all the variation of f between x_1 and x_2, and we're just getting to the bottom line, the net change in y between x_1 and x_2. This is a gross oversimplification of f, but it's much easier to see and compute than *all* of f between those x's.

Sometimes we want to use a **formula** to describe the behavior of a function f. Typically, this is an algebraic expression, but it need not always be so. And it is frequently, though not always, in one variable. In such a case, we usually denote the variable by x, and we write $f(x)$ to denote that the function depends on the variable x, which is appropriately called the **independent variable**. Since to each a in the domain of f we assign one, and only one, b, such that $f(a) = b$, we say that b depends on a. If we have a formula for f, then we assign a variable y to x, satisfying $y = f(x)$, and we say y depends on x, or y is the **dependent variable**.

For example,

$$f(x) = x^2$$

is the parabola. The formula here is x^2. This notation is shorthand for the relation

$$\{(x, y) \mid y = x^2,\ x \in \mathbb{R}\}$$

i.e.

$$\{(a, a^2) \mid a \in \mathbb{R}\}$$

We often want to graph the function, and this just means plotting the relation in the plane $A \times B$ (usually the x-y plane). An easy way to graph a function g is by transforming the graph of a known function f in such a way as to get g. To do this, we need to know that the graph of a function can be **transformed** in six different ways:

1. **Vertical translation** by k:
$$f(x) \mapsto f(x) \pm k$$
If $+$, the shift is upward, if $-$, it's downward.
2. **Horizontal translation** by h:
$$f(x) \mapsto f(x \pm h)$$
If $+$, the shift is to the left, if $-$, it's to the right.
3. **Reflection about the x-axis**:
$$f(x) \mapsto -f(x)$$
4. **Reflection about the y-axis**:
$$f(x) \mapsto f(-x)$$
5. **Vertical stretch or compression**:
$$f(x) \mapsto af(x)$$
where $a > 0$.
   (a) If $0 < a < 1$, this is a vertical compression.
   (b) If $1 < a$, this is a vertical stretch.
6. **Horizontal stretch or compression**:
$$f(x) \mapsto f(ax)$$
where $a > 0$.
   (a) If $0 < a < 1$, this is a horizontal stretch.
   (b) If $1 < a$, this is a horizontal compression.

A function is **even** if it's symmetric about the y-axis, in which case it must satisfy $f(x) = f(-x)$. The parabola $f(x) = x^2$ is an example. A function f is **odd** if it is symmetric about the origin, that is if it satisfies $f(x) = -f(x)$. It is entirely possible that a function is neither even nor odd, for example $f(x) = x^2 + x^3$.

A function may be **increasing**, **decreasing**, or **constant**. It's increasing if

$$x_1 < x_2 \implies f(x_1) < f(x_2)$$

that is if f preserves the order relations of points, or, graphically, if it goes up from left to right. It is decreasing if

$$x_1 < x_2 \implies f(x_1) > f(x_2)$$

that is if f reverses the order relations of points, or, graphically, if it goes down from left to right. It is constant if it never increases or decreases. Graphically, this would be a horizontal line.