
 College Algebra Final, 7/2/10

1. Factor the polynomial $p(x) = 3x^5 - 13x^4 + 13x^3 + 9x^2 - 16x + 4$ completely, then sketch a graph of it. Make sure to plot the x - and y -intercepts. (10 points)

Solution: By the rational roots theorem, if p/q is to be a root, then p divides 4 and q divides 3, so the possibilities for p/q are

$$\pm 1, \pm 2, \pm 4, \pm \frac{1}{3}, \pm \frac{2}{3}, \pm \frac{4}{3}$$

As usual, we start with the easiest, 1, and try that with synthetic division:

$$1 \begin{array}{r|rrrrrr} 3 & -13 & 13 & 9 & -16 & 4 \\ & & 3 & -10 & 3 & 12 & -4 \\ \hline & 3 & -10 & 3 & 12 & -4 & 0 \end{array}$$

So 1 is a root, and therefore $(x - 1)$ is a factor. Let's see if 1 is a double root. We'll try synthetic division of the *quotient* that we just got, $3x^4 - 10x^3 + 3x^2 + 12x - 4$, by $(x + 1)$:

$$1 \begin{array}{r|rrrrr} 3 & -10 & 3 & 12 & -4 \\ & & 3 & -7 & -4 & 8 \\ \hline & 3 & -7 & -4 & 8 & 4 \end{array}$$

No. Next, we try it with -1 :

$$-1 \begin{array}{r|rrrrr} 3 & -10 & 3 & 12 & -4 \\ & & -3 & 13 & -16 & 4 \\ \hline & 3 & -13 & 16 & -4 & 0 \end{array}$$

Yes! Let's try -1 with *this* quotient, to see if it's a double root:

$$-1 \begin{array}{r|rrrr} 3 & -13 & 16 & -4 \\ & & -3 & 16 & -32 \\ \hline & 3 & -16 & 32 & -36 \end{array}$$

No. We've exhausted 1 and -1 , having seen they work no more than once each, so we move on to 2:

$$2 \begin{array}{r|rrrr} 3 & -13 & 16 & -4 \\ & & 6 & -14 & 4 \\ \hline & 3 & -7 & 2 & 0 \end{array}$$

Yes! Let's summarize what we've found so far:

$$3x^5 - 13x^4 + 13x^3 + 9x^2 - 16x + 4 = (x - 1)(x + 1)(x - 2)(3x^2 - 7x + 2)$$

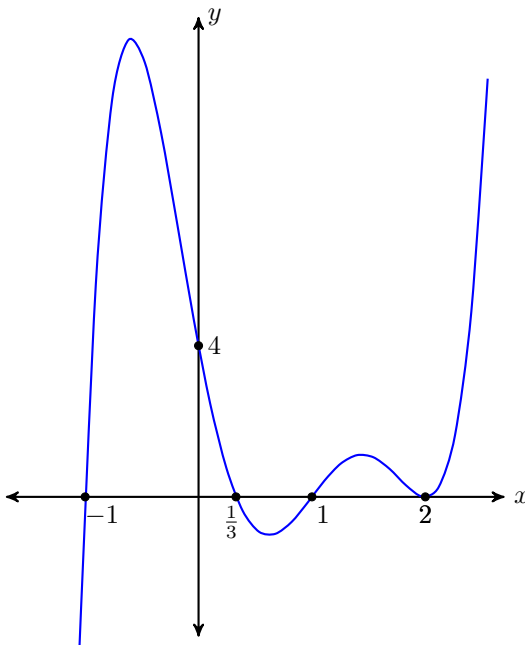
The last quotient, $3x^2 - 7x + 2$, factors directly:

$$3x^2 - 7x + 2 = (3x - 1)(x - 2)$$

Altogether, then, we've found:

$$\begin{aligned} 3x^5 - 13x^4 + 13x^3 + 9x^2 - 16x + 4 &= (x - 1)(x + 1)(x - 2)(3x - 1)(x - 2) \\ &= \boxed{(x - 1)(x + 1)(x - 2)^2(3x - 1)} \end{aligned}$$

Now that we have it factored, it's easy to graph. Since it's degree is 5, an odd number, and since the leading of $p(x)$ coefficient is 3, we know it goes up from left to right. Since the roots 1, -1, and $\frac{1}{3}$ have odd multiplicity (one each), $p(x)$ goes through those points, and it bounces off 2, which has even multiplicity (two). It's y -intercept is $(0, p(0)) = (0, 4)$ (just plug in 0):



Note, this isn't quite drawn to scale.

2. Compute the product $(3 + 2\sqrt{10})^3$. (5 points)

$$\begin{aligned} (3 + 2\sqrt{10})^3 &= (3 + 2\sqrt{10})^2(3 + 2\sqrt{10}) \\ &= (3^2 + 2 \cdot 3 \cdot 2\sqrt{10} + (2\sqrt{10})^2)(3 + 2\sqrt{10}) \\ &= (49 + 12\sqrt{10})(3 + 2\sqrt{10}) \\ &= 49 \cdot 3 + 3 \cdot 12\sqrt{10} + 49 \cdot 2\sqrt{10} + (12\sqrt{10})(2\sqrt{10}) \\ &= 147 + 240 + 36\sqrt{10} + 98\sqrt{10} \\ &= \boxed{387 + 134\sqrt{10}} \end{aligned}$$

3. Solve the equation $e^{2x}e^1 = \frac{e^3}{e^{3x-8}}$. (5 points)

Solution: Just use exponent rules to rewrite each side,

$$e^{2x+1} = e^{3-(3x-8)} = e^{11-3x}$$

Since the exponential function is one-to-one, we may equate the exponents:

$$2x + 1 = 11 - 3x$$

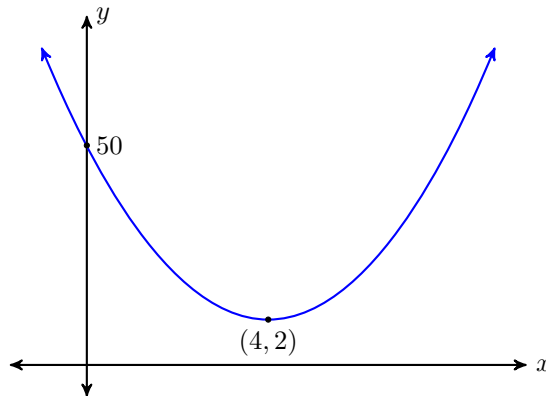
Solving for x shows that $x = 2$.

4. Graph the quadratic function $f(x) = 3x^2 - 24x + 50$. Make sure to plot the vertex and any x - and y -intercepts. (5 points)

Solution: Complete the square first,

$$\begin{aligned} f(x) &= 3x^2 - 24x + 50 \\ &= 3(x^2 - 8x) + 50 \\ &= 3(x - 8x + 16) + 50 - 48 \\ &= 3(x - 4)^2 + 2 \end{aligned}$$

From this we know that the vertex is at $(4, 2)$. Since the leading term 3 is positive, the parabola opens upward. It doesn't have any x -intercepts, since it lies entirely above the x -axis. However, it does have a y -intercept, at $(0, f(0)) = (0, 50)$:



Note, this isn't drawn to scale.

5. Write the exponential expression $8^2 = 64$ in logarithmic form. (5 points)

$$\log_8 64 = 2$$

6. Simplify the expression $\left(\frac{16x^8y^{12}b^{24}}{a^{-4}b^8x^{-8}}\right)^{1/4}$. (5 points)

$$2x^4y^3ab^4$$

7. Let $f(x) = x^2 - 3x + 2$ and let $g(x) = 3^x$.

(a) Find $(f \circ g)(x)$. (5 points)

$$\begin{aligned} (f \circ g)(x) &= f(g(x)) \\ &= f(3^x) \\ &= (3^x)^2 - 3(3^x) + 2 \\ &= \boxed{3^{2x} - 3^{x+1} + 2} \end{aligned}$$

(b) Solve $(f \circ g)(x) = 0$. [Hint: Use u -substitution first.] (5 points)

Let $u = 3^x = g(x)$ and note that $(f \circ g)(x) = f(g(x)) = f(u) = u^2 - 3u + 2$. This factors,

$$f(u) = (u - 2)(u - 1)$$

Thus,

$$0 = (f \circ g)(x) = (u - 2)(u - 1) = (3^x - 2)(3^x - 1)$$

By the zero product property, this means that one of the factors must be zero. Set each one of them equal to zero and solve for x :

$$3^x - 2 = 0$$

$$3^x = 2$$

$$\boxed{x = \log_3 2}$$

$$3^x - 1 = 0$$

$$3^x = 1$$

$$\boxed{x = \log_3 1 = 0}$$

8. Use long division to write $\frac{p(x)}{d(x)} = \frac{4x^3 - 5x^2 + 2x + 7}{x^2 - 1}$ in the form $\frac{p(x)}{d(x)} = q(x) + \frac{r(x)}{d(x)}$. (5 points)

Long division gives

$$\begin{array}{r} \overline{4x - 5} \\ x^2 - 1 \overline{) 4x^3 - 5x^2 + 2x + 7} \\ \underline{-4x^3} + 4x \\ - 5x^2 + 6x + 7 \\ \underline{5x^2} - 5 \\ 6x + 2 \end{array}$$

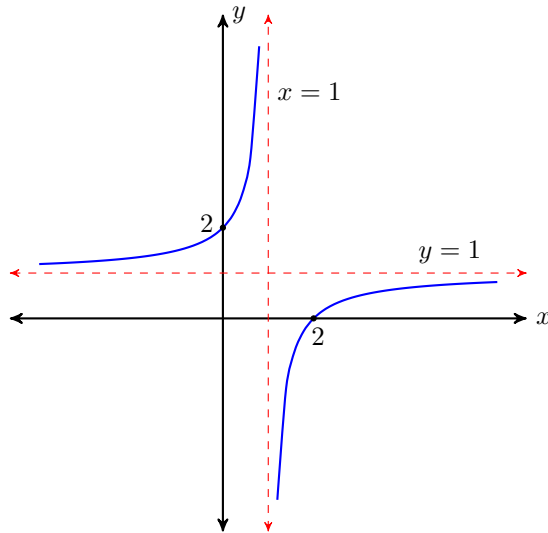
Note, this LaTeX package for long division makes it look like you're adding negatives instead of subtracting on each line, but that's just because it's programmed that way. In any case, you find that $q(x) = 4x - 5$ and $r(x) = 6x + 2$, so

$$\boxed{\frac{4x^3 - 5x^2 + 2x + 7}{x^2 - 1} = 4x - 5 + \frac{6x + 2}{x^2 - 1}}$$

9. Graph $\frac{x - 2}{x - 1}$. Make sure to plot any x - and y -intercepts, and label any horizontal and vertical asymptotes. (5 points)

Solution: Since $x - 1$ is in the denominator, we know x cannot equal 1, and so $x = 1$ is a vertical asymptote. Since we have degree one polynomials on top and bottom, and since their leading coefficients are 1, we know that a horizontal asymptote occurs at $y = \frac{1}{1} = 1$. Next, plug in $x = 0$, to get $(0, 2)$ for the y -intercept. Let the function equal 0 to find the x -intercept

(2, 0). All this tells you what the graph must look like (if you're still not sure, plot a couple of points to see):



Note that if we had performed long division we would have gotten:

$$\begin{array}{r} \overline{) x-2} \\ -x+1 \\ \hline -1 \end{array}$$

so that

$$\frac{x-2}{x-1} = 1 - \frac{1}{x-1}$$

which means that the graph of $\frac{1}{x}$ was translated to the right 1, up 1, and flipped over. This might have made graphing this easier. In any case, it provides another way of doing it.

10. Consider the polynomial $p(x) = x^4 - 7x^2 + 9$. Circle the intervals in which $p(x)$ has a root. (5 points)

$[-3, -2]$

$[-2, -1]$

$[-1, 0]$

$[0, 1]$

$[1, 2]$

$[2, 3]$

The way we get these are as follows:

$$f(-3) = 27 > 0 > -3 = f(-2)$$

so by the Intermediate Value Theorem we know there is a root between -3 and -2 . Likewise,

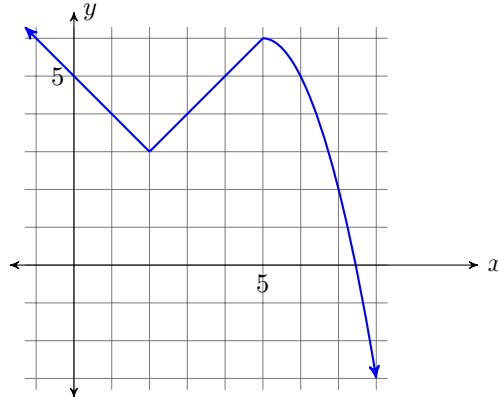
$$f(-2) = -3 < 0 < 3 = f(-1)$$

so we know there is a root between -2 and -1 . Etc. The other two cases are identical, since $f(3) = f(-3)$ and $f(2) = f(-2)$, because the function is even. Once you've found these four, it isn't necessary to look further, since a fourth degree polynomial can have a max of four roots.

11. Find the equation of the line through the points $(0, 1)$ and $(1, 3)$ and write it in slope-intercept form. (5 points)

$$y = 2x + 1$$

12. Find the equation for the function whose graph is given below: (5 points)



Solution:
$$f(x) = \begin{cases} |x - 2| + 3, & \text{if } x \leq 5 \\ -(x - 5)^2 + 6, & \text{if } x > 5 \end{cases}$$

13. Let $f(x) = \sqrt{x + 2}$ and $g(x) = x^2 - 1$.

- (a) What is the domain of $(f \circ g)(x)$? (5 points)

Since $(f \circ g)(x) = \sqrt{(x^2 - 1) + 2} = \sqrt{x^2 + 1}$, the domain will be all x 's such that $x^2 + 1 \geq 0$, which means all x 's such that $x^2 \geq -1$. But this is always true, because the square of any real number is always nonnegative. Therefore the domain is all $\mathbb{R} = (-\infty, \infty)$.

- (b) What is the domain of $(f + g)(x)$? (5 points)

Subce $(f + g)(x) = \sqrt{x + 2} + x^2 - 1$, the domain will be all x 's such that $x + 2 \geq 0$, i.e. $[-2, \infty)$.

14. Three times the first of two consecutive even integers plus four times the second is equal to eleven times the first. Find the integers. (5 points)

$$2 \text{ and } 4$$

15. Solve the inequality $|x + 2| \leq 7$. (5 points)

We can do this by cases, or we can do it by a transformation. Let's do it by cases first: case 1, $x + 2 \leq 7$, gives $x \leq 5$, and case 2, $-(x + 2) \leq 7$, gives $-x - 2 \leq 7$, so $-9 \leq x$. Together, this means all x 's in the interval

$$[-9, 5]$$

The other way to do this is to notice that the solution to $|x| \leq 7$ is $[-7, 7]$, and then $|x + 2| \leq 7$ is merely a transformation of this solution, namely by a left shift by 2.

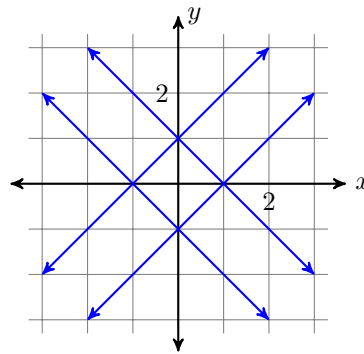
Extra Credit:

1. Graph the set of points $\{(x, y) : |x| + |y| = 1\}$. [Hint: Think about what $|x| + |y| = 1$ means. How many possible cases are there?] (5 points)

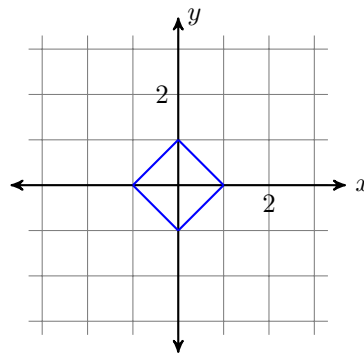
First, notice that there are four cases. To each of $|x|$ and $|y|$ there correspond two cases, one positive and one negative, and we can combine them to give four total possibilities:

$$\begin{aligned}x + y &= 1 \\ -x + y &= 1 \\ x - y &= 1 \\ -x - y &= 1\end{aligned}$$

These are four lines, $y = -x + 1$, $y = x + 1$, $y = x - 1$ and $y = -x - 1$. Graphing these is easy:



But we're not done. Because the answer can't be all these points. For consider the point $(1, 2)$ on the line $y = x + 1$. Is it true that $1 + 2 = 1$? No. Thus, everything outside of the diamond isn't correct, for the same reason. The set we want is just the diamond:



2. Solve $5^{x+1} = 6^{2x}$. [Note the different bases, 5 and 6.] (5 points)

There are two ways to do this. The first: take log base 5 of both sides, and use the power rule,

$$x + 1 = \log_5(6^{2x}) = 2x \log_5 6$$

Then solve this for x :

$$x - 2x \log_5 6 = -1$$

so $x(1 - 2 \log_5 6) = -1$, or

$$x = -\frac{1}{\log_5 6}$$

Alternatively, take log base 6 of both sides:

$$2x = \log_6(5^{x+1}) = (x+1) \log_6 5 = x \log_6 5 + \log_6 5$$

so

$$2x - x \log_6 5 = \log_6 5$$

Thus, $x(2 - \log_6 5) = \log_6 5$, so

$$x = \frac{\log_6 5}{2 - \log_6 5}$$