MATH 2300-001
Final Exam Review Solutions:

1. $-\frac{d}{dx}\left[\cosh(\operatorname{sech}(2x))\right] = \sinh\left(\operatorname{sech}(2x)\right)\operatorname{sech}(2x)\tanh(2x)2.$

2.

$$\begin{array}{r}
x+2 \\
x^2-2x+1 \overline{\big)\; x^3 \qquad\qquad\qquad} \\
\underline{-x^3+2x^2\ -x} \\
2x^2\ -x \\
\underline{-2x^2+4x-2} \\
3x-2
\end{array}$$

$$\begin{aligned}
\int \frac{x^3}{x^2-2x+1}\,dx &= \int x+2+\frac{3x-2}{(x-1)^2}\,dx \\
&= \frac{1}{2}x^2+2x+\int \frac{3}{x-1}+\frac{1}{(x-1)^2}\,dx \\
&= \frac{1}{2}x^2+2x+3\ln|x-1|-\frac{1}{x-1}+C
\end{aligned}$$

3.

$$\begin{array}{ccc}
x^4 & + & e^{-x} \\
4x^3 & - & -e^{-x} \\
12x^2 & + & e^{-x} \\
24x & - & -e^{-x} \\
24 & + & e^{-x} \\
0 & & -e^{-x}
\end{array}$$

$$\int_0^1 x^4e^{-x}\,dx = \left[-e^{-x}\left(x^4+4x^3+12x^2+24x+24\right)\right]_0^1 = 24-65e^{-1}.$$

4.

$$\int_0^1 e^{x^2}\,dx = \sum_{k=0}^{\infty}\int_0^1 \frac{x^{2k}}{k!}\,dx = \sum_{k=0}^{\infty}\left[\frac{x^{2k+1}}{k!(2k+1)}\right]_0^1 = \sum_{k=0}^{\infty}\frac{1}{k!(2k+1)}.$$

5.

$$\begin{aligned}
\int_{\pi/4}^{\pi/3} \tan^5 x\sec^4 x\,dx &= \int_{\pi/4}^{\pi/3} \tan^5 x(\tan^2 x+1)\sec^2 x\,dx \\
&= \int_1^{\sqrt{3}} u^5(u^2+1)\,du \\
&= \left[\frac{1}{8}u^8+\frac{1}{6}u^6\right]_1^{\sqrt{3}} \\
&= \frac{81}{8}+\frac{27}{6}-\frac{1}{8}-\frac{1}{6} \\
&= \frac{43}{3}.
\end{aligned}$$

6.

$$\int_1^\infty \frac{1}{x^2\sqrt{x^2-1}}\,dx = \int_0^{\frac{\pi}{2}} \frac{\sec\theta\tan\theta}{\sec^2\theta\tan\theta}\,d\theta = \int_0^{\frac{\pi}{2}} \cos\theta\,d\theta = [\sin\theta]_0^{\frac{\pi}{2}} = 1.$$

7. Find the general solution to the differential equation $(1+x)y' + y = \sqrt{x}$.

$$\begin{aligned}(1+x)y' + y &= \sqrt{x}\\ \frac{d}{dx}\left[(1+x)y\right] &= \sqrt{x}\\ (1+x)y &= \frac{2}{3}x^{\frac{3}{2}} + C\\ y(x) &= \frac{1}{1+x}\left(\frac{2}{3}x^{\frac{3}{2}} + C\right).\end{aligned}$$

8. Solve the initial value problem $y'' - 2\sqrt{2}y' + 2y = 0, y(0) = 0, y'(0) = 5$.

$$\begin{aligned}0 &= k^2 - 2\sqrt{2}k + 2 = \left(k - \sqrt{2}\right)^2\\ y(x) &= (c_1 + c_2 x)e^{\sqrt{2}x}\\ 0 &= c_1\\ y'(x) &= (c_2 + c_2\sqrt{2}x)e^{\sqrt{2}x}\\ 5 &= c_2\\ y(x) &= 5xe^{\sqrt{2}x}.\end{aligned}$$

9. Determine if the sequence converges. If it does, find its limit.

(a) $\left\{\frac{(-4)^n}{n!}\right\}_{n=0}^{\infty}$

$\sum_{n=0}^{\infty}\frac{(-4)^n}{n!} = e^{-4}$, and so, is a convergent series. Thus, $\lim_{n\to\infty}\frac{(-4)^n}{n!} = 0$.

(b) $\left\{\frac{1-(-1)^n}{\sqrt{n}}\right\}_{n=1}^{\infty}$

$0 \le \frac{1-(-1)^n}{\sqrt{n}} \le \frac{2}{\sqrt{n}} \underset{n\to\infty}{\longrightarrow} 0$. So, by the squeeze theorem, $\lim_{n\to\infty}\frac{1-(-1)^n}{\sqrt{n}} = 0$.

10. Determine if the series diverges, converges conditionally, or converges absolutely. If the series converges, find its sum.

(a) $\sum_{k=0}^{\infty}\frac{1}{(k+1)(k+3)}$

$$\begin{aligned}\sum_{k=0}^{n}\frac{1}{(k+1)(k+3)} &= \frac{1}{2}\sum_{k=0}^{n}\frac{1}{k+1} - \frac{1}{k+3}\\ &= \frac{1}{2}\sum_{k=0}^{n}\frac{1}{k+1} - \frac{1}{2}\sum_{k=2}^{n+2}\frac{1}{k+1}\\ &= \frac{1}{2}\left(1 + \frac{1}{2} - \frac{1}{n+1} - \frac{1}{n+2}\right)\\ &\underset{n\to\infty}{\longrightarrow} \frac{3}{4}.\end{aligned}$$

Thus, $\sum_{k=0}^{\infty}\frac{1}{(k+1)(k+3)}$ converges absolutely to $\frac{3}{4}$.

(b)

$$\sum_{k=1}^{\infty} \frac{k+1}{k^2+(-1)^k k+1} \geq \sum_{k=1}^{\infty} \frac{k}{3k^2} = \sum_{k=1}^{\infty} \frac{1}{3k}$$

Thus, the series diverges by the Direct Comparison Test to a divergent p-series.

(c) $\displaystyle\sum_{k=1}^{\infty} \frac{10^k}{k4^{2k+1}} = -\frac{1}{4}\sum_{k=1}^{\infty} \frac{(-1)^{k+1}\left(-\frac{10}{16}\right)^k}{k} = -\frac{1}{4}\ln\left(1-\frac{5}{8}\right) = \frac{1}{4}\ln\left(\frac{8}{3}\right).$

The convergence is absolute since the terms of the series are nonnegative.

(d) $\displaystyle\sum_{k=0}^{\infty} \frac{(-5)^k}{k!} = e^{-5}.$

The convergence is absolute since the Maclaurin series for e^x converges absolutely everywhere.

11. Find the interval of convergence for the power series $\displaystyle\sum_{k=0}^{\infty} \frac{(k+1)x^{2k-1}}{3^k}$

$$\lim_{k\to\infty} \left| \frac{(k+2)x^{2k+1}}{3^{k+1}} \frac{3^k}{(k+1)x^{2k-1}} \right| = \lim_{k\to\infty} \frac{k+2}{k+1}\frac{x^2}{3} = \frac{x^2}{3} < 1 \Rightarrow -\sqrt{3} < x < \sqrt{3}.$$

$$x = \pm\sqrt{3}\colon \sum_{k=0}^{\infty} \frac{(k+1)x^{2k-1}}{3^k} = \sum_{k=0}^{\infty} \frac{k+1}{\pm\sqrt{3}}$$

This series diverges by the divergence test. Thus, the interval of convergence is $(-\sqrt{3}, \sqrt{3})$.

12. Find the Taylor series for $f(x) = \frac{1}{1-x}$ expanded around $x = 5$.

$$\frac{1}{1-x} = \frac{1}{-4-(x-5)} = -\frac{1}{4}\frac{1}{1-\frac{x-5}{-4}} = -\frac{1}{4}\sum_{k=0}^{\infty}\left(\frac{x-5}{-4}\right)^k = \sum_{k=0}^{\infty} \frac{(-1)^{k+1}}{4^{k+1}}(x-5)^k.$$

13. Approximate $\cos(\frac{1}{2})$ to 2 decimal places using the Maclaurin series for $\cos(x)$.

$$\cos\left(\frac{1}{2}\right) = \sum_{k=0}^{\infty} \frac{(-1)^k}{(2k)!}\left(\frac{1}{2}\right)^{2k} = \sum_{k=0}^{\infty} \frac{(-1)^k}{(2k)!4^k}$$

$$\begin{aligned} \frac{1}{(2n+2)!4^{n+1}} &\leq .005 \\ \frac{200}{4^{n+1}} &\leq (2n+2)! \\ \frac{25}{2^{2n-1}} &\leq (2n+2)! \end{aligned}$$

$$n = 0 : 50 \not\leq 2 \qquad n = 1 : 12.5 \leq 24.$$

$$\cos\left(\frac{1}{2}\right) \approx 1 - \frac{1}{2!\cdot 4} = \frac{7}{8}.$$

14. Using series, prove that $\cos^2\theta = \frac{1}{2}\left(1+\cos(2\theta)\right).$

$$\begin{aligned}
2\cos^2\theta &= 2\left(1-\frac{\theta^2}{2!}+\frac{\theta^4}{4!}-\frac{\theta^6}{6!}+\cdots\right)\cdot\left(1-\frac{\theta^2}{2!}+\frac{\theta^4}{4!}-\frac{\theta^6}{6!}+\cdots\right)\\
&= 2\left(1-\left(\frac{1}{2!}+\frac{1}{2!}\right)\theta^2+\left(\frac{1}{4!}+\frac{1}{2!2!}+\frac{1}{4!}\right)\theta^4-\left(\frac{1}{6!}+\frac{1}{4!2!}+\frac{1}{2!}4!+\frac{1}{6!}\right)\theta^6+\cdots\right)\\
&= 2\left(\frac{1}{0!}-\frac{2}{2!}\theta^2+\frac{8}{4!}\theta^4-\frac{32}{6!}\theta^6+\cdots\right)\\
&= 1+\left(\frac{1}{0!}-\frac{(2\theta)^2}{2!}+\frac{(2\theta)^4}{4!}-\frac{(2\theta)^6}{6!}+\cdots\right)\\
&= 1+\cos(2\theta).
\end{aligned}$$

Thus, $\cos^2\theta = \frac{1}{2}\left(1+\cos(2\theta)\right).$

15. Convert the polar equation $\sin(2\theta) = 1$ to Cartesian coordinates.

$$\begin{aligned}
\sin(2\theta) &= 1\\
2\theta &= \frac{\pi}{2}+2\pi k\\
\theta &= \frac{\pi}{4}+\pi k\\
\tan\theta &= 1\\
\frac{y}{x} &= 1\\
y &= x.
\end{aligned}$$

16. Find the length of $r = \theta$ from $\theta = 0$ to $\theta = \pi$.

$$\begin{aligned}
L &= \int_0^{\pi}\sqrt{1^2+\theta^2}\,d\theta\\
&= \int_0^{\tan^{-1}\pi}\sqrt{1+\tan^2 u}\sec^2 u\,du\\
&= \int_0^{\tan^{-1}\pi}\sec u\sec^2 u\,du\\
&= \left[\sec u\tan u\right]_0^{\tan^{-1}\pi} - \int_0^{\tan^{-1}\pi}\sec u\tan^2 u\,du\\
&= \left[\sec u\tan u\right]_0^{\tan^{-1}\pi} + \int_0^{\tan^{-1}\pi}\sec u\,du - \int_0^{\tan^{-1}\pi}\sec^3 u\,du\\
&= \frac{1}{2}\left[\sec u\tan u+\ln|\sec u+\tan u|\right]_0^{\tan^{-1}\pi}\\
&= \frac{1}{2}\left(\pi\sqrt{1+\pi^2}+\ln|\sqrt{1+\pi^2}+\pi\right).
\end{aligned}$$

17. Find the area between the loops of the limaçon $r = 1 + \sqrt{2}\sin\theta$.

$$
\begin{aligned}
A &= 2\int_{\frac{-\pi}{4}}^{\frac{\pi}{2}} \frac{1}{2}(1+\sqrt{2}\sin\theta)^2\,d\theta - 2\int_{\frac{-\pi}{2}}^{\frac{-\pi}{4}} \frac{1}{2}(1+\sqrt{2}\sin\theta)^2\,d\theta \\
&= \int_{\frac{-\pi}{4}}^{\frac{\pi}{2}} 1 + 2\sqrt{2}\sin\theta + 2\sin^2\theta\,d\theta - \int_{\frac{-\pi}{2}}^{\frac{-\pi}{4}} 1 + 2\sqrt{2}\sin\theta + 2\sin^2\theta\,d\theta \\
&= \int_{\frac{-\pi}{4}}^{\frac{\pi}{2}} 2 + 2\sqrt{2}\sin\theta - \cos(2\theta)\,d\theta - \int_{\frac{-\pi}{2}}^{\frac{-\pi}{4}} 2 + 2\sqrt{2}\sin\theta - \cos(2\theta)\,d\theta \\
&= \left[2\theta - 2\sqrt{2}\cos\theta - \frac{1}{2}\sin(2\theta)\right]_{\frac{-\pi}{4}}^{\frac{\pi}{2}} - \left[2\theta - 2\sqrt{2}\cos\theta - \frac{1}{2}\sin(2\theta)\right]_{\frac{-\pi}{2}}^{\frac{-\pi}{4}} \\
&= (\pi - 0 - 0) - \left(-\frac{\pi}{2} - 2\sqrt{2}\frac{\sqrt{2}}{2} + \frac{1}{2}\right) - \left(-\frac{\pi}{2} - 2\sqrt{2}\frac{\sqrt{2}}{2} + \frac{1}{2}\right) + (-\pi - 0 - 0) \\
&= -\left(-\pi - 4 + 1\right) \\
&= \pi + 3.
\end{aligned}
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