SO(3), SU(2) and Some Related Results

Define the group isomorphism

$$\boxed{\varphi : (\{\pm 1\}, \cdot) \longrightarrow (\mathbb{Z}_2, +)}$$

by

$$\boxed{\varphi(x) := \tfrac{1}{2}\ln(e^{1-x})}$$

i.e.

$$\varphi(1) = \tfrac{1}{2}\ln(e^{1-1}) = 0$$

$$\varphi(-1) = \tfrac{1}{2}\ln(e^{1-(-1)}) = 1$$

(we note that $\varphi(1\cdot 1) = \varphi(1) = 0 = 0+0 = \varphi(1)+\varphi(1)$,

$\varphi(1\cdot(-1)) = \varphi(-1) = 1 = 0+1 = \varphi(1)+\varphi(-1)$

$\varphi((-1)\cdot(-1)) = \varphi(1) = 0 = 1+1 \pmod 2$

$= \varphi(-1)+\varphi(-1)$

so φ is indeed a homomorphism, & $\therefore$ an isomorphism.)

Claim 1 $\boxed{SO(2) \approx U(1) \approx S^1 \ (\text{diffeo.})}$ (DK)

pf: The group homomorphism

$$\varphi : (\mathbb{R}, +) \longrightarrow (SO(2), \circ)$$

$$\varphi(t) := \begin{pmatrix} \cos t & -\sin t \\ \sin t & \cos t \end{pmatrix}$$

has kernel

$$\ker\varphi = \varphi^{-1}(I_2) = \{t \in \mathbb{R} \mid \cos t = 1, \sin t = 0\}$$

$$= 2\pi\mathbb{Z}$$

so by the 1st isom thm.

$$SO(2) = \operatorname{im}\varphi \cong \mathbb{R}/\ker\varphi$$
$$= \mathbb{R}/2\pi\mathbb{Z}$$
$$\approx S^1$$
$$\approx U(1)$$

QED

• The case of $SO(3)$

Recall that $\mathbb{R}^3$ equipped with the cross product $\times \in \mathrm{Hom}^2_{skew}(\mathbb{R}^3;\mathbb{R}^3)$ is a Lie algebra, which has a representation on $\mathbb{R}^3$,

$$\rho : (\mathbb{R}^3, \times) \longrightarrow \big(\overset{\substack{so(3)\\ \downarrow}}{sl(3)} = \mathrm{Skew}_3(\mathbb{R}), \circ\big)$$

$$\rho\begin{pmatrix} a \\ b \\ c \end{pmatrix} := \begin{pmatrix} 0 & -c & b \\ c & 0 & -a \\ -b & a & 0 \end{pmatrix}$$

and the action (the $\mathbb{R}^3$-module action on $\mathbb{R}^3$, the representation space) is taking the cross product,

$$\mathbb{R}^3 \times \mathbb{R}^3 \longrightarrow \mathbb{R}^3$$

$$(\vec{x}, \vec{y}) \longmapsto \rho(\vec{x})\,\vec{y} = \rho\begin{pmatrix} a \\ b \\ c \end{pmatrix} \cdot \begin{pmatrix} d \\ e \\ f \end{pmatrix}$$

$$= \begin{pmatrix} 0 & -c & b \\ c & 0 & -a \\ -b & a & 0 \end{pmatrix}\begin{pmatrix} d \\ e \\ f \end{pmatrix} = \begin{pmatrix} bf - ce \\ -(af - cd) \\ ae - bd \end{pmatrix}$$

where $M(\mathbb{R}^n) = Sym_n^+(\mathbb{R})$ is the set of pos. def. symmetric matrices, or equiv. the (reps. of) metrics on $\mathbb{R}^n$!

The exponential is a diffeomorphism btw. $Sym_n(\mathbb{R})$ & $Sym_n^+(\mathbb{R})$:

$$\boxed{\exp : Sym_n(\mathbb{R}) \xrightarrow{\approx} Sym_n^+(\mathbb{R})}$$

but it is only surjective on $Skew_n(\mathbb{R})$,

$$\boxed{\exp : Skew_n(\mathbb{R}) \overset{(\text{but not } \approx)}{\twoheadrightarrow} SO(n)}$$

since θ & $\theta + 2n\pi$ give the same value

since, e.g.

$$\exp\begin{pmatrix} 0 & -\theta \\ \theta & 0 \end{pmatrix} = \sum_{k=0}^{\infty} \frac{1}{k!} \begin{pmatrix} 0 & -\theta \\ \theta & 0 \end{pmatrix}^k$$

$$= \sum_{k=0}^{\infty} \frac{\theta^k}{k!} \begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix}^k$$

but $\begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix}^2 = -\begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}$, $\begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix}^3 = \begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix}$, so by the 1st. isom thm. we have, again,

$$\boxed{SO(2) = \operatorname{im}\varphi \cong \mathbb{R}/\ker\varphi = \mathbb{R}/2\pi\mathbb{Z} \approx S^1}$$

- Let us now consider the case of $SO(3)$ and $\exp : so(3) = \mathrm{Skew}_3(\mathbb{R}) \cong \mathbb{R}^3 \longrightarrow SO(3)$. Let $\vec{x} = \begin{pmatrix} a \\ b \\ c \end{pmatrix} \in \mathbb{R}^3$ and $\rho(\vec{x}) = \begin{pmatrix} 0 & -c & b \\ c & 0 & -a \\ -b & a & 0 \end{pmatrix}$ its representation in $so(3)$, and note

$$\rho(\vec{x})^2 = \begin{pmatrix} 0 & -c & b \\ c & 0 & -a \\ -b & a & 0 \end{pmatrix}^2 = \begin{pmatrix} -(b^2+c^2) & ab & ac \\ ab & -(a^2+c^2) & bc \\ ac & bc & -(a^2+b^2) \end{pmatrix}$$

&

$$\rho(\vec{x})^3 = \rho(\vec{x})\rho(\vec{x})^2 = \begin{pmatrix} abc-abc & c(a^2+c^2)+b^2c & -bc^2-b(a^2+b^2) \\ -a^2c-c(b^2+c^2) & abc-abc & ac^2+a(a^2+b^2) \\ a^2b+b(b^2+c^2) & -ab^2-a(a^2+c^2) & abc-abc \end{pmatrix}$$

$$= \begin{pmatrix} 0 & c\|\vec{x}\|^2 & -b\|\vec{x}\|^2 \\ -c\|\vec{x}\|^2 & 0 & a\|\vec{x}\|^2 \\ b\|\vec{x}\|^2 & a\|\vec{x}\|^2 & 0 \end{pmatrix}$$

$$= -\|\vec{x}\|^2\rho(\vec{x})$$

so $\rho(\vec{x})^4 = -\|\vec{x}\|^2\rho(\vec{x})^2$, $\rho(\vec{x})^5 = \|\vec{x}\|^4\rho(\vec{x})$, $\rho(\vec{x})^6 = \|\vec{x}\|\rho(\vec{x})^2$, so

$$\boxed{e^{\rho(\vec{x})} = I + \left(\frac{1}{\|\vec{x}\|}\sin\|\vec{x}\|\right)\rho(\vec{x}) + \frac{1}{\|\vec{x}\|^2}\left(1-\cos\|\vec{x}\|\right)\rho(\vec{x})^2}$$

For similar reasons, $\exp : so(3) = \mathrm{Skew}_3(\mathbb{R}) \to SO(3)$ is surjective, but not injective,

$$\boxed{\exp : \mathrm{Skew}_3(\mathbb{R}) \xrightarrow{\text{(but not 1-1)}} SO(3)}$$

and similarly exp is a diffeomorphism fr. $\mathrm{Sym}_3(\mathbb{R})$ to $\mathrm{Sym}_3^+(\mathbb{R}) = M(\mathbb{R}^3)$,

$$\boxed{\exp : \mathrm{Sym}_3(\mathbb{R}) \xrightarrow{\approx} \mathrm{Sym}_3^+(\mathbb{R}) = M(\mathbb{R}^3)}$$

To see what happens with $\mathrm{Skew}_3(\mathbb{R})$, we use the identification

$$\boxed{\begin{array}{c} \mathrm{Skew}_3(\mathbb{R}) \xrightarrow{\sim} \mathbb{R}^3 \\ \begin{pmatrix} 0 & -c & b \\ c & 0 & -a \\ -b & a & 0 \end{pmatrix} \longmapsto \begin{pmatrix} a \\ b \\ c \end{pmatrix} \end{array}}$$

pf: Let $A \in SO(3)$. By the spectral theorem for normal operators (& particularly for orthog. operators) & the fact that $\deg c_A(x) = 3$, $c_A(x) \in \mathbb{R}[x]$, we know that $c_A(x)$ has a real root, $\lambda_1 \in \mathbb{R}$, so

$$c_A(x) = (x - \lambda_1)(x^2 + bx + c)$$

Since $A \in SO(3)$, if $\vec{x} \in \mathbb{R}^3$ is an eigenvector then $A\vec{x} = \lambda_1 \vec{x}$, & ~~$(A\vec{x})\cdot(A\vec{x}) = \vec{x}$~~

$$\begin{aligned}
\lambda_1^2(\vec{x}\cdot\vec{x}) &= (\lambda_1\vec{x})\cdot(\lambda_1\vec{x}) \\
&= (A\vec{x})\cdot(A\vec{x}) \\
&= (A\vec{x})^T(A\vec{x}) \\
&= \vec{x}^T A^T A \vec{x} \\
&= \vec{x}^T\vec{x} && \text{since } A^TA = I \\
&= \vec{x}\cdot\vec{x} && \text{bec. } A \in SO(3)
\end{aligned}$$

so $\lambda_1^2 = 1 \implies \lambda_1 = \pm 1$. Since $A \in SO(3)$ (& not just $O(3)$), $\lambda_1 = 1$, & $\therefore$

$$E_{\lambda_1} = \{\vec{x} \in \mathbb{R}^3 \mid A\vec{x} = \vec{x}\} = \text{axis of rotation}$$

so, by Rodrigues' formula,

$$\boxed{\begin{aligned} \exp(\rho(\vec{v}_1)) &= I + \sin\theta\,\rho(\vec{v}_1) + (1-\cos\theta)\,\rho(\vec{v}_1)^2 \\ &= A \\ &= M^{-1}\rho(\vec{v}_1)\,M \end{aligned}}$$

which shows the surjectivity of $\varphi = \exp\circ\rho$ fr. B_π to $SO(3)$ (actually, by ~~standardizing $\vec{v}_1, \vec{v}_2, \vec{v}_3 \in B_\pi$ we used B_1, but $B_1 \approx B_\pi$ by scaling~~ using $\boxed{\theta = \|\vec{x}\|/\pi}$ we used B_1, but by rescaling we have our claim.).

QED

Claim 2 $\varphi := \exp\circ\rho$ restricted to B_π° is a diffeomorphism bet. B_π° & the open, dense subset of rotations in $SO(3)$ through $\theta \in [0,\pi)$.

pf: Take $\theta = \|\vec{x}\| < \pi$, i.e. $\theta = \|\vec{x}\| \in [0,\pi)$, i.e. $\vec{x} \in B_\pi^\circ$, above.

QED

$$= I + \frac{\sin(\|\vec{x}\| - 2\pi)}{\|\vec{x}\|}\rho(\vec{x}) + \frac{1-\cos(\|\vec{x}\| - 2\pi)}{\|\vec{x}\|^2}\rho(\vec{x})^2$$

$$\cancel{\in I + \frac{\sin\|\vec{x}\|}{\|\vec{x}\|} + \frac{1-\cos\|\vec{x}\|}{\|\vec{x}\|^2}}$$

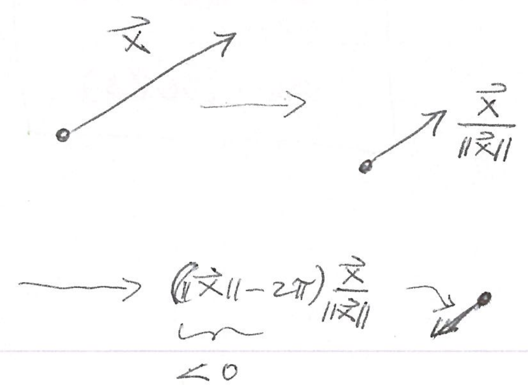
$$= I + \frac{\sin\|\vec{x}\|}{\|\vec{x}\|}\rho(\vec{x}) + \frac{1-\cos\|\vec{x}\|}{\|\vec{x}\|^2}\rho(\vec{x})^2$$

$$= R_{\|\vec{x}\|}$$

$$= \varphi(\vec{x})$$

Thus, $\vec{x} \in B^\circ_{2\pi}$ & $\left(1 - \frac{2\pi}{\|\vec{x}\|}\right)\vec{x} = (\|\vec{x}\| - 2\pi)\frac{\vec{x}}{\|\vec{x}\|} \in B^\circ_{2\pi}$

are mapped to the same element $R_{\|\vec{x}\|}$ by φ:



QED

Thm. SO(n) is path connected, hence connected.

pf: From the spectral theorem we know that any $A \in SO(n)$ can be put into a canonical form of the type

$$PAP^{-1} = \bigoplus_{i=1}^{k} \begin{pmatrix} \cos\theta_i & -\sin\theta_i \\ \sin\theta_i & \cos\theta_i \end{pmatrix} \oplus (-I) \oplus I$$

even size

& since $-I$ is even size, we can replace in pairs the -1's by $R_\pi = \begin{pmatrix} \cos\pi & -\sin\pi \\ \sin\pi & \cos\pi \end{pmatrix}$, thereby eliminating $-I$ & absorbing it into the 1st part:

$$PAP^{-1} = \bigoplus_{i=1}^{m} \begin{pmatrix} \cos\theta_i & -\sin\theta_i \\ \sin\theta_i & \cos\theta_i \end{pmatrix} \oplus I_{n-2m}$$

$$= \bigoplus_{i=1}^{m} \exp\begin{pmatrix} 0 & -\theta_i \\ \theta_i & 0 \end{pmatrix} \oplus I_{n-2m}$$

Now define the path in SO(n)