

SO(3), SU(2) and Some  
Related Results

Define the group isomorphism

$$\varphi : (\{\pm 1\}, \cdot) \rightarrow (\mathbb{Z}_2, +)$$

$$\varphi(x) := \frac{1}{2} \ln(e^{1-x})$$

by  
↓

i.e.

$$\varphi(1) = \frac{1}{2} \ln(e^{1-1}) = 0$$

$$\varphi(-1) = \frac{1}{2} \ln(e^{1-(-1)}) = 1$$

(we note that  $\varphi(1 \cdot 1) = \varphi(1) = 0 = 0 + 0 = \varphi(1) + \varphi(1)$ ,

~~$\varphi(1 \cdot (-1)) = \varphi(-1) = 1 = 0 + 1 = \varphi(1) + \varphi(-1)$~~

$$\varphi(1 \cdot (-1)) = \varphi(-1) = 1 = 0 + 1 = \varphi(1) + \varphi(-1)$$

$$\varphi((-1) \cdot (-1)) = \varphi(1) = 0 = 1 + 1 \pmod{2}$$

$$= \varphi(-1) + \varphi(-1)$$

so  $\varphi$  is indeed a homomorphism, & ∴ an isomorphism.)

Claim 1  $SO(2) \approx U(1) \approx S^1$  (diffeo.) (DK)

pf: The group homomorphism

$$\varphi: (\mathbb{R}, +) \rightarrow (SO(2), \cdot)$$

$$\varphi(t) := \begin{pmatrix} \cos t & -\sin t \\ \sin t & \cos t \end{pmatrix}$$

has kernel

$$\ker \varphi = \varphi^{-1}(I_2) = \left\{ t \in \mathbb{R} \mid \begin{array}{l} \cos t = 1, \\ \sin t = 0 \end{array} \right\}$$

$$= 2\pi\mathbb{Z}$$

so by the 1st isom. thm.

$$SO(2) = \text{im } \varphi \cong \mathbb{R} / \ker \varphi$$

$$= \mathbb{R} / 2\pi\mathbb{Z}$$

$$\cong S^1$$

$$\cong U(1)$$

QED

## The case of $SO(3)$

Recall that  $\mathbb{R}^3$  equipped with the cross product  $\times \in \text{Hom}_{\text{skew}}^2(\mathbb{R}^3; \mathbb{R}^3)$  is a Lie algebra, which has a representation on  $\mathbb{R}^3$ ,

$$\rho: (\mathbb{R}^3, \times) \rightarrow \text{so}(3) = \text{Skew}_3(\mathbb{R}, \circ)$$

$$\rho \begin{pmatrix} a \\ b \\ c \end{pmatrix} := \begin{pmatrix} 0 & -c & b \\ c & 0 & -a \\ -b & a & 0 \end{pmatrix}$$

and the action (the  $\mathbb{R}^3$ -module action on  $\mathbb{R}^3$ , the representation space) is taking the cross product,

$$\mathbb{R}^3 \times \mathbb{R}^3 \longrightarrow \mathbb{R}^3$$

$$(\vec{x}, \vec{y}) \longmapsto \rho(\vec{x})\vec{y} = \rho \begin{pmatrix} a \\ b \\ c \end{pmatrix} \cdot \begin{pmatrix} d \\ e \\ f \end{pmatrix}$$

$$= \begin{pmatrix} 0 & -c & b \\ c & 0 & -a \\ -b & a & 0 \end{pmatrix} \begin{pmatrix} d \\ e \\ f \end{pmatrix} = \begin{pmatrix} bf - ce \\ -(af - cd) \\ ae - bd \end{pmatrix}$$

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where  $M(\mathbb{R}^n) = \text{Sym}_n^+(\mathbb{R})$  is the set of pos. def. symmetric matrices, or equiv. the (rep.s of.) metrics on  $\mathbb{R}^n$ .

The exponential is a diffeomorphism bet.  $\text{Sym}_n(\mathbb{R})$  &  $\text{Sym}_n^+(\mathbb{R})$ :

~~exp:  $\text{Sym}_n(\mathbb{R}) \rightarrow \text{Sym}_n^+(\mathbb{R})$~~

$$\text{exp} : \text{Sym}_n(\mathbb{R}) \xrightarrow{\approx} \text{Sym}_n^+(\mathbb{R})$$

but it is only surjective on  $\text{Skew}_n(\mathbb{R})$ ,

$$\text{exp} : \text{Skew}_n(\mathbb{R}) \xrightarrow{\text{(but not } \approx \text{)}} \text{SO}(n)$$

since  $\theta \neq \theta + 2n\pi$  give the same value

since, e.g.

$$\begin{aligned} \text{exp} \begin{pmatrix} 0 & -\theta \\ \theta & 0 \end{pmatrix} &= \sum_{k=0}^{\infty} \frac{1}{k!} \begin{pmatrix} 0 & -\theta \\ \theta & 0 \end{pmatrix}^k \\ &= \sum_{k=0}^{\infty} \frac{\theta^k}{k!} \begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix}^k \end{aligned}$$

$$\text{but } \begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix}^2 = - \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}, \quad \begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix}^3 = \begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix},$$

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so by the 1st. iso. thm. we have, again,

$$SO(2) = \text{im } \varphi \cong \mathbb{R}/\ker \varphi = \mathbb{R}/2\pi\mathbb{Z} \approx S^1$$

• Let us now consider the case of  $SO(3)$  and

$\exp: \mathfrak{so}(3) = \text{Skew}_3(\mathbb{R}) \cong \mathbb{R}^3 \rightarrow SO(3)$ . Let

$\vec{x} = \begin{pmatrix} a \\ b \\ c \end{pmatrix} \in \mathbb{R}^3$  and  $\rho(\vec{x}) = \begin{pmatrix} 0 & -c & b \\ c & 0 & -a \\ -b & a & 0 \end{pmatrix}$  its representation in  $\mathfrak{so}(3)$ , and note

$$\rho(\vec{x})^2 = \begin{pmatrix} 0 & -c & b \\ c & 0 & -a \\ -b & a & 0 \end{pmatrix}^2 = \begin{pmatrix} -(b^2+c^2) & ab & ac \\ ab & -(a^2+b^2) & bc \\ ac & bc & -(a^2+b^2) \end{pmatrix}$$

and

$$\begin{aligned} \rho(\vec{x})^3 &= \rho(\vec{x})\rho(\vec{x})^2 = \begin{pmatrix} abc-abc & c(a^2+c^2)+b^2c & -bc^2-b(b^2+c^2) \\ -a^2c-c(b^2+c^2) & abc-abc & ac^2+a(a^2+b^2) \\ a^2b+b(b^2+c^2) & -ab^2-a(a^2+c^2) & abc-abc \end{pmatrix} \\ &= \begin{pmatrix} 0 & c\|\vec{x}\|^2 & -b\|\vec{x}\|^2 \\ -c\|\vec{x}\|^2 & 0 & a\|\vec{x}\|^2 \\ b\|\vec{x}\|^2 & a\|\vec{x}\|^2 & 0 \end{pmatrix} \end{aligned}$$

$$= -\|\vec{x}\|^2 \rho(\vec{x})$$

so  $\rho(\vec{x})^4 = -\|\vec{x}\|^2 \rho(\vec{x})^2$ ,  $\rho(\vec{x})^5 = \|\vec{x}\|^4 \rho(\vec{x})$ ,  $\rho(\vec{x})^6 = \|\vec{x}\|^4 \rho(\vec{x})^2$ ,  
 $\rho(\vec{x})^7 = -\|\vec{x}\|^6 \rho(\vec{x})$ ,  $\rho(\vec{x})^8 = \|\vec{x}\|^6 \rho(\vec{x})^2$ ,  $\rho(\vec{x})^9 = -\|\vec{x}\|^8 \rho(\vec{x})$ ,  $\rho(\vec{x})^{10} = \|\vec{x}\|^8 \rho(\vec{x})^2$ ,  $\rho(\vec{x})^{11} = -\|\vec{x}\|^{10} \rho(\vec{x})$ ,  $\rho(\vec{x})^{12} = \|\vec{x}\|^{10} \rho(\vec{x})^2$ ,  $\rho(\vec{x})^{13} = -\|\vec{x}\|^{12} \rho(\vec{x})$ ,  $\rho(\vec{x})^{14} = \|\vec{x}\|^{12} \rho(\vec{x})^2$ ,  $\rho(\vec{x})^{15} = -\|\vec{x}\|^{14} \rho(\vec{x})$ ,  $\rho(\vec{x})^{16} = \|\vec{x}\|^{14} \rho(\vec{x})^2$ ,  $\rho(\vec{x})^{17} = -\|\vec{x}\|^{16} \rho(\vec{x})$ ,  $\rho(\vec{x})^{18} = \|\vec{x}\|^{16} \rho(\vec{x})^2$ ,  $\rho(\vec{x})^{19} = -\|\vec{x}\|^{18} \rho(\vec{x})$ ,  $\rho(\vec{x})^{20} = \|\vec{x}\|^{18} \rho(\vec{x})^2$ ,  $\rho(\vec{x})^{21} = -\|\vec{x}\|^{20} \rho(\vec{x})$ ,  $\rho(\vec{x})^{22} = \|\vec{x}\|^{20} \rho(\vec{x})^2$ ,  $\rho(\vec{x})^{23} = -\|\vec{x}\|^{22} \rho(\vec{x})$ ,  $\rho(\vec{x})^{24} = \|\vec{x}\|^{22} \rho(\vec{x})^2$ ,  $\rho(\vec{x})^{25} = -\|\vec{x}\|^{24} \rho(\vec{x})$ ,  $\rho(\vec{x})^{26} = \|\vec{x}\|^{24} \rho(\vec{x})^2$ ,  $\rho(\vec{x})^{27} = -\|\vec{x}\|^{26} \rho(\vec{x})$ ,  $\rho(\vec{x})^{28} = \|\vec{x}\|^{26} \rho(\vec{x})^2$ ,  $\rho(\vec{x})^{29} = -\|\vec{x}\|^{28} \rho(\vec{x})$ ,  $\rho(\vec{x})^{30} = \|\vec{x}\|^{28} \rho(\vec{x})^2$ ,  $\rho(\vec{x})^{31} = -\|\vec{x}\|^{30} \rho(\vec{x})$ ,  $\rho(\vec{x})^{32} = \|\vec{x}\|^{30} \rho(\vec{x})^2$ ,  $\rho(\vec{x})^{33} = -\|\vec{x}\|^{32} \rho(\vec{x})$ ,  $\rho(\vec{x})^{34} = \|\vec{x}\|^{32} \rho(\vec{x})^2$ ,  $\rho(\vec{x})^{35} = -\|\vec{x}\|^{34} \rho(\vec{x})$ ,  $\rho(\vec{x})^{36} = \|\vec{x}\|^{34} \rho(\vec{x})^2$ ,  $\rho(\vec{x})^{37} = -\|\vec{x}\|^{36} \rho(\vec{x})$ ,  $\rho(\vec{x})^{38} = \|\vec{x}\|^{36} \rho(\vec{x})^2$ ,  $\rho(\vec{x})^{39} = -\|\vec{x}\|^{38} \rho(\vec{x})$ ,  $\rho(\vec{x})^{40} = \|\vec{x}\|^{38} \rho(\vec{x})^2$ ,  $\rho(\vec{x})^{41} = -\|\vec{x}\|^{40} \rho(\vec{x})$ ,  $\rho(\vec{x})^{42} = \|\vec{x}\|^{40} \rho(\vec{x})^2$ ,  $\rho(\vec{x})^{43} = -\|\vec{x}\|^{42} \rho(\vec{x})$ ,  $\rho(\vec{x})^{44} = \|\vec{x}\|^{42} \rho(\vec{x})^2$ ,  $\rho(\vec{x})^{45} = -\|\vec{x}\|^{44} \rho(\vec{x})$ ,  $\rho(\vec{x})^{46} = \|\vec{x}\|^{44} \rho(\vec{x})^2$ ,  $\rho(\vec{x})^{47} = -\|\vec{x}\|^{46} \rho(\vec{x})$ ,  $\rho(\vec{x})^{48} = \|\vec{x}\|^{46} \rho(\vec{x})^2$ ,  $\rho(\vec{x})^{49} = -\|\vec{x}\|^{48} \rho(\vec{x})$ ,  $\rho(\vec{x})^{50} = \|\vec{x}\|^{48} \rho(\vec{x})^2$ ,  $\rho(\vec{x})^{51} = -\|\vec{x}\|^{50} \rho(\vec{x})$ ,  $\rho(\vec{x})^{52} = \|\vec{x}\|^{50} \rho(\vec{x})^2$ ,  $\rho(\vec{x})^{53} = -\|\vec{x}\|^{52} \rho(\vec{x})$ ,  $\rho(\vec{x})^{54} = \|\vec{x}\|^{52} \rho(\vec{x})^2$ ,  $\rho(\vec{x})^{55} = -\|\vec{x}\|^{54} \rho(\vec{x})$ ,  $\rho(\vec{x})^{56} = \|\vec{x}\|^{54} \rho(\vec{x})^2$ ,  $\rho(\vec{x})^{57} = -\|\vec{x}\|^{56} \rho(\vec{x})$ ,  $\rho(\vec{x})^{58} = \|\vec{x}\|^{56} \rho(\vec{x})^2$ ,  $\rho(\vec{x})^{59} = -\|\vec{x}\|^{58} \rho(\vec{x})$ ,  $\rho(\vec{x})^{60} = \|\vec{x}\|^{58} \rho(\vec{x})^2$ ,  $\rho(\vec{x})^{61} = -\|\vec{x}\|^{60} \rho(\vec{x})$ ,  $\rho(\vec{x})^{62} = \|\vec{x}\|^{60} \rho(\vec{x})^2$ ,  $\rho(\vec{x})^{63} = -\|\vec{x}\|^{62} \rho(\vec{x})$ ,  $\rho(\vec{x})^{64} = \|\vec{x}\|^{62} \rho(\vec{x})^2$ ,  $\rho(\vec{x})^{65} = -\|\vec{x}\|^{64} \rho(\vec{x})$ ,  $\rho(\vec{x})^{66} = \|\vec{x}\|^{64} \rho(\vec{x})^2$ ,  $\rho(\vec{x})^{67} = -\|\vec{x}\|^{66} \rho(\vec{x})$ ,  $\rho(\vec{x})^{68} = \|\vec{x}\|^{66} \rho(\vec{x})^2$ ,  $\rho(\vec{x})^{69} = -\|\vec{x}\|^{68} \rho(\vec{x})$ ,  $\rho(\vec{x})^{70} = \|\vec{x}\|^{68} \rho(\vec{x})^2$ ,  $\rho(\vec{x})^{71} = -\|\vec{x}\|^{70} \rho(\vec{x})$ ,  $\rho(\vec{x})^{72} = \|\vec{x}\|^{70} \rho(\vec{x})^2$ ,  $\rho(\vec{x})^{73} = -\|\vec{x}\|^{72} \rho(\vec{x})$ ,  $\rho(\vec{x})^{74} = \|\vec{x}\|^{72} \rho(\vec{x})^2$ ,  $\rho(\vec{x})^{75} = -\|\vec{x}\|^{74} \rho(\vec{x})$ ,  $\rho(\vec{x})^{76} = \|\vec{x}\|^{74} \rho(\vec{x})^2$ ,  $\rho(\vec{x})^{77} = -\|\vec{x}\|^{76} \rho(\vec{x})$ ,  $\rho(\vec{x})^{78} = \|\vec{x}\|^{76} \rho(\vec{x})^2$ ,  $\rho(\vec{x})^{79} = -\|\vec{x}\|^{78} \rho(\vec{x})$ ,  $\rho(\vec{x})^{80} = \|\vec{x}\|^{78} \rho(\vec{x})^2$ ,  $\rho(\vec{x})^{81} = -\|\vec{x}\|^{80} \rho(\vec{x})$ ,  $\rho(\vec{x})^{82} = \|\vec{x}\|^{80} \rho(\vec{x})^2$ ,  $\rho(\vec{x})^{83} = -\|\vec{x}\|^{82} \rho(\vec{x})$ ,  $\rho(\vec{x})^{84} = \|\vec{x}\|^{82} \rho(\vec{x})^2$ ,  $\rho(\vec{x})^{85} = -\|\vec{x}\|^{84} \rho(\vec{x})$ ,  $\rho(\vec{x})^{86} = \|\vec{x}\|^{84} \rho(\vec{x})^2$ ,  $\rho(\vec{x})^{87} = -\|\vec{x}\|^{86} \rho(\vec{x})$ ,  $\rho(\vec{x})^{88} = \|\vec{x}\|^{86} \rho(\vec{x})^2$ ,  $\rho(\vec{x})^{89} = -\|\vec{x}\|^{88} \rho(\vec{x})$ ,  $\rho(\vec{x})^{90} = \|\vec{x}\|^{88} \rho(\vec{x})^2$ ,  $\rho(\vec{x})^{91} = -\|\vec{x}\|^{90} \rho(\vec{x})$ ,  $\rho(\vec{x})^{92} = \|\vec{x}\|^{90} \rho(\vec{x})^2$ ,  $\rho(\vec{x})^{93} = -\|\vec{x}\|^{92} \rho(\vec{x})$ ,  $\rho(\vec{x})^{94} = \|\vec{x}\|^{92} \rho(\vec{x})^2$ ,  $\rho(\vec{x})^{95} = -\|\vec{x}\|^{94} \rho(\vec{x})$ ,  $\rho(\vec{x})^{96} = \|\vec{x}\|^{94} \rho(\vec{x})^2$ ,  $\rho(\vec{x})^{97} = -\|\vec{x}\|^{96} \rho(\vec{x})$ ,  $\rho(\vec{x})^{98} = \|\vec{x}\|^{96} \rho(\vec{x})^2$ ,  $\rho(\vec{x})^{99} = -\|\vec{x}\|^{98} \rho(\vec{x})$ ,  $\rho(\vec{x})^{100} = \|\vec{x}\|^{98} \rho(\vec{x})^2$ .

so

$$e^{p(\vec{x})} = I + \left( \frac{1}{\|\vec{x}\|} \sin \|\vec{x}\| \right) p(\vec{x}) + \frac{1}{\|\vec{x}\|^2} (1 - \cos \|\vec{x}\|) p(\vec{x})^2$$

For similar reasons,  $\exp: \mathfrak{so}(3) = \text{Skew}_3(\mathbb{R}) \rightarrow \text{SO}(3)$  is surjective, but not injective,

$$\exp: \text{Skew}_3(\mathbb{R}) \xrightarrow{\text{(not 1-1)}} \text{SO}(3)$$

and similarly  $\exp$  is a diffeomorphism fr.  $\text{Sym}_3(\mathbb{R})$  to  $\text{Sym}_3^+(\mathbb{R}) = \mathcal{M}(\mathbb{R}^3)$ ,

$$\exp: \text{Sym}_3(\mathbb{R}) \xrightarrow{\sim} \text{Sym}_3^+(\mathbb{R}) = \mathcal{M}(\mathbb{R}^3)$$

To see what happens with  $\text{Skew}_3(\mathbb{R})$ , we use the identification

$$\text{Skew}_3(\mathbb{R}) \xrightarrow{\sim} \mathbb{R}^3$$

$$\begin{pmatrix} 0 & -c & b \\ c & 0 & -a \\ -b & a & 0 \end{pmatrix} \mapsto \begin{pmatrix} a \\ b \\ c \end{pmatrix}$$

pf: Let  $A \in SO(3)$ . By the spectral theorem for normal operators (and particularly for orthog. operators) and the fact that  $\deg c_A(x) = 3$ ,  $c_A(x) \in \mathbb{R}[x]$ , we know that  $c_A(x)$  has a real root,  $\lambda_1 \in \mathbb{R}$ , so

$$c_A(x) = (x - \lambda_1)(x^2 + bx + c)$$

Since  $A \in SO(3)$ , if  $\vec{x} \in \mathbb{R}^3$  is an eigenvector then  $A\vec{x} = \lambda_1\vec{x}$ , and  ~~$A^T(A\vec{x}) = \vec{x}$~~

~~$$\lambda_1^2(\vec{x} \cdot \vec{x}) = (\lambda_1\vec{x}) \cdot (\lambda_1\vec{x})$$~~

$$= (A\vec{x}) \cdot (A\vec{x})$$

$$= (A\vec{x})^T (A\vec{x})$$

$$= \vec{x}^T A^T A \vec{x}$$

$$= \vec{x}^T \vec{x} \quad \text{since } A^T A = I$$

$$= \vec{x} \cdot \vec{x} \quad \text{but } A \in SO(3)$$

so  $\lambda_1^2 = 1 \Rightarrow \lambda_1 = \pm 1$ . Since

$A \in SO(3)$  (and not just  $O(3)$ ),  $\lambda_1 = 1$ , and  $\therefore$

$E_{\lambda_1} = \{\vec{x} \in \mathbb{R}^3 \mid A\vec{x} = \vec{x}\} = \text{axis of rotation}$

so, by Rodrigues' formula,

$$\begin{aligned}
 \exp(\rho(\vec{v}_i)) &= I + \sin \theta \rho(\vec{v}_i) + (1 - \cos \theta) \rho(\vec{v}_i)^2 \\
 &= \cancel{A} \\
 &= \Gamma^{-1} \rho(\vec{v}_i) \Gamma.
 \end{aligned}$$

which shows the surjectivity of  $\varphi = \exp \circ \rho$  fr.  $B_\pi$  to  $SO(3)$  (actually, by ~~standard def of  $\vec{v}_i, \vec{v}_j \in B_\pi$~~   
~~as  $B_\pi \approx S^1$ , but  $B_\pi \approx B_\pi$  by scaling~~).

using  $\theta = \|\vec{x}\|/\pi$  we used  $B_\pi$ , but by rescaling we have our claim. QED

Claim 2  $\varphi := \exp \circ \rho$  restricted to  $B_\pi^0$  is a diffeomorphism bet.  $B_\pi^0$  & the open, dense subset of rotations in  $SO(3)$  through  $\theta \in [0, \pi)$ .

pf: Take  $\theta = \|\vec{x}\| < \pi$ , i.e.  $\theta = \|\vec{x}\| \in [0, \pi)$ ,  
 i.e.  $\vec{x} \in B_\pi^0$ , above. QED



$$= I + \frac{\sin(\|\vec{x}\| - 2\pi)}{\|\vec{x}\|} \rho(\vec{x}) + \frac{1 - \cos(\|\vec{x}\| - 2\pi)}{\|\vec{x}\|^2} \rho(\vec{x})^2$$

~~$$= I + \frac{\sin \|\vec{x}\|}{\|\vec{x}\|} \rho(\vec{x}) + \frac{1 - \cos \|\vec{x}\|}{\|\vec{x}\|^2} \rho(\vec{x})^2$$~~

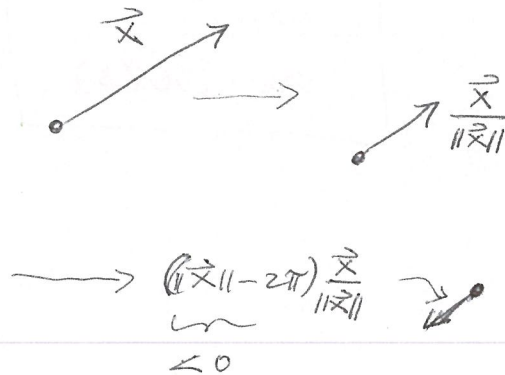
$$= I + \frac{\sin \|\vec{x}\|}{\|\vec{x}\|} \rho(\vec{x}) + \frac{1 - \cos \|\vec{x}\|}{\|\vec{x}\|^2} \rho(\vec{x})^2$$

$$= R_{\|\vec{x}\|}$$

$$= \varphi(\vec{x})$$

Thus,  $\vec{x} \in B_{2\pi}^0$  &  $(1 - \frac{2\pi}{\|\vec{x}\|}) \vec{x} = (\|\vec{x}\| - 2\pi) \frac{\vec{x}}{\|\vec{x}\|} \in B_{2\pi}^0$

are mapped to the same element  $R_{\|\vec{x}\|}$  by  $\varphi$ :



QED

Thm.  $SO(n)$  is path connected, hence connected.

pf: From the spectral theorem we know that any  $A \in SO(n)$  can be put into a canonical form of the type

$$PAP^{-1} = \bigoplus_{i=1}^k \begin{pmatrix} \cos \theta_i & -\sin \theta_i \\ \sin \theta_i & \cos \theta_i \end{pmatrix} \oplus \underbrace{\begin{pmatrix} -1 & \\ & -1 \end{pmatrix}}_{\substack{\text{even} \\ \text{size}}} \oplus I$$

if since  $-I$  is even size, we can replace ~~it~~ in pairs the  $-1$ 's by  $R_\pi = \begin{pmatrix} \cos \pi & -\sin \pi \\ \sin \pi & \cos \pi \end{pmatrix}$ , thereby eliminating  $-I$  & absorbing it into the 1st part:

$$PAP^{-1} = \bigoplus_{i=1}^m \begin{pmatrix} \cos \theta_i & -\sin \theta_i \\ \sin \theta_i & \cos \theta_i \end{pmatrix} \oplus I_{n-2m}$$

$$\text{~~it~~} = \bigoplus_{i=1}^m \exp \begin{pmatrix} 0 & -\theta_i \\ \theta_i & 0 \end{pmatrix} \oplus I_{n-2m}$$

Now define the path in  $SO(n)$

~~it~~