Eigenvalues & Eigenvectors II

Thm. 1 For any $A \in M_n(F)$, $F = \mathbb{R}$ or $\mathbb{C}$,

$$\boxed{A \in GL(n,F) \iff \text{there are elementary matrices } E_1, \dots, E_k \in \mathcal{E}_n \text{ with } E_k E_{k-1} \cdots E_1 (A \mid I_n) = (I_n \mid B)}$$

In this case

$$\boxed{B = E_k E_{k-1} \cdots E_1 = A^{-1}}$$

Remark: This says, also, that $\operatorname{rref} A = I_n$ $\iff A \in GL(n,F)$, so $\boxed{A \sim_R I_n \iff A \in GL(n,F)}$

$\boxtimes$

ex. 1 $\quad A = \begin{pmatrix} 1 & 3 & 3 \\ 3 & 5 & 3 \\ 3 & 3 & 1 \end{pmatrix} \in M_3(\mathbb{R}).$

$$(A \mid I_3) = \left(\begin{array}{ccc|ccc} 1 & 3 & 3 & 1 & 0 & 0 \\ 3 & 5 & 3 & 0 & 1 & 0 \\ 3 & 3 & 1 & 0 & 0 & 1 \end{array}\right)$$

$E_1 = \begin{pmatrix} 1 & 0 & 0 \\ -3 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix} \in \mathcal{E}_3^{III}$:

$$E_1(A \mid I_3) = \begin{pmatrix} 1 & 0 & 0 \\ -3 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix}\left(\begin{array}{ccc|ccc} 1 & 3 & 3 & 1 & 0 & 0 \\ 3 & 5 & 3 & 0 & 1 & 0 \\ 3 & 3 & 1 & 0 & 0 & 1 \end{array}\right) = \left(\begin{array}{ccc|ccc} 1 & 3 & 3 & 1 & 0 & 0 \\ 0 & -4 & -6 & -3 & 1 & 0 \\ 3 & 3 & 1 & 0 & 0 & 1 \end{array}\right)$$

$E_2 = \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ -3 & 0 & 1 \end{pmatrix} \in \mathcal{E}_3^{III}$:

$$E_2E_1(A \mid I_3) = \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ -3 & 0 & 1 \end{pmatrix}\left(\begin{array}{ccc|ccc} 1 & 3 & 3 & 1 & 0 & 0 \\ 0 & -4 & -6 & -3 & 1 & 0 \\ 3 & 3 & 1 & 0 & 0 & 1 \end{array}\right) = \left(\begin{array}{ccc|ccc} 1 & 3 & 3 & 1 & 0 & 0 \\ 0 & -4 & -6 & -3 & 1 & 0 \\ 0 & -6 & -8 & -3 & 0 & 1 \end{array}\right)$$

$E_3 = \begin{pmatrix} 1 & 0 & 0 \\ 0 & -\frac{1}{2} & 0 \\ 0 & 0 & 1 \end{pmatrix} \in \mathcal{E}_3^{II}$:

$$E_3E_2E_1(A \mid I_3) = \begin{pmatrix} 1 & 0 & 0 \\ 0 & -\frac{1}{2} & 0 \\ 0 & 0 & 1 \end{pmatrix}\left(\begin{array}{ccc|ccc} 1 & 3 & 3 & 1 & 0 & 0 \\ 0 & -4 & -6 & -3 & 1 & 0 \\ 0 & -6 & -8 & -3 & 0 & 1 \end{array}\right) = \left(\begin{array}{ccc|ccc} 1 & 3 & 3 & 1 & 0 & 0 \\ 0 & 2 & 3 & \frac{3}{2} & -\frac{1}{2} & 0 \\ 0 & -6 & -8 & -3 & 0 & 1 \end{array}\right)$$

$E_4 = \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & -\frac{1}{2} \end{pmatrix} \in \mathcal{E}_3^{II}$:

$$E_4E_3E_2E_1(A \mid I_3) = \left(\begin{array}{ccc|ccc} 1 & 3 & 3 & 1 & 0 & 0 \\ 0 & 2 & 3 & 3/2 & -1/2 & 0 \\ 0 & 3 & 4 & 3/2 & 0 & -1/2 \end{array}\right)$$

$E_5 = \begin{pmatrix} 1 & 0 & -1 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix} \in \mathcal{E}_3^{III}$:

$$E_5 \cdots E_1(A \mid I_3) = \begin{pmatrix} 1 & 0 & -1 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix}\left(\begin{array}{ccc|ccc} 1 & 3 & 3 & 1 & 0 & 0 \\ 0 & 2 & 3 & 3/2 & -1/2 & 0 \\ 0 & 3 & 4 & 3/2 & 0 & -1/2 \end{array}\right) = \left(\begin{array}{ccc|ccc} 1 & 0 & -1 & -1/2 & 0 & 1/2 \\ 0 & 2 & 3 & 3/2 & -1/2 & 0 \\ 0 & 3 & 4 & 3/2 & 0 & -1/2 \end{array}\right)$$

$$E_6 = \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & -3/2 & 1 \end{pmatrix} : \quad E_6 \cdots E_1 (A \mid I_3)$$

$$\in \mathcal{E}_3^{III} \qquad\qquad = \left(\begin{array}{ccc|ccc} 1 & 0 & -1 & -1/2 & 0 & 1/2 \\ 0 & 2 & 3 & 3/2 & -1/2 & 0 \\ 0 & 0 & -1/2 & -3/4 & 3/4 & -1/2 \end{array}\right)$$

$$E_7 = \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & -2 \end{pmatrix} : \quad E_7 \cdots E_1 (A \mid I_3) = \left(\begin{array}{ccc|ccc} 1 & 0 & -1 & -1/2 & 0 & 1/2 \\ 0 & 2 & 3 & 3/2 & -1/2 & 0 \\ 0 & 0 & 1 & 3/2 & -3/2 & 1 \end{array}\right)$$

$$\in \mathcal{E}_3^{II}$$

$$E_8 = \begin{pmatrix} 1 & 0 & 1 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix} : \quad E_8 \cdots E_1 = \left(\begin{array}{ccc|ccc} 1 & 0 & 0 & 1 & -3/2 & 3/2 \\ 0 & 2 & 3 & 3/2 & -1/2 & 0 \\ 0 & 0 & 1 & 3/2 & -3/2 & 1 \end{array}\right)$$

$$\in \mathcal{E}_3^{III}$$

Remark: $\det.\begin{pmatrix} 1 & 0 & 0 \\ 0 & 2 & 3 \\ 0 & 0 & 1 \end{pmatrix} = 1 \cdot 2 \cdot 1 = 2$, & since $E_1, E_2, E_5, E_6, E_8 \in \mathcal{E}_3^{III}$

& $E_3, E_4, E_7 \in \mathcal{E}_3^{II}$

E_3, E_4: by $-1/2$; E_7: by -2

$$2 = \det(E_8 \cdots E_1 A) = \underbrace{\left(-\tfrac{1}{2}\right)\cancel{\left(-\tfrac{1}{2}\right)}}_{\text{fr. } E_3, E_4}\underbrace{\cancel{(-2)}}_{\text{fr. } E_7} \det A = -\tfrac{1}{2} \det A$$

we get $\boxed{\det A = -4}$

$\boxtimes$

Anyway, continuing,

$$E_9 = \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & -3 \\ 0 & 0 & 1 \end{pmatrix} : \quad E_9 \cdots E_1 (A \mid I_3)$$

$$= \left(\begin{array}{ccc|ccc} 1 & 0 & 0 & 1 & -3/2 & 3/2 \\ 0 & 2 & 0 & -3 & 4 & -3 \\ 0 & 0 & 1 & 3/2 & -3/2 & 1 \end{array}\right)$$

$$E_{10} = \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1/2 & 0 \\ 0 & 0 & 1 \end{pmatrix} : \quad E_{10} \cdots E_1 (A \mid I_3)$$

$$= \left(\begin{array}{ccc|ccc} 1 & 0 & 0 & 1 & -3/2 & 3/2 \\ 0 & 1 & 0 & -3/2 & 2 & -3/2 \\ 0 & 0 & 1 & 3/2 & -3/2 & 1 \end{array}\right)$$

$$= (I_3 \mid B) \text{ or } (I_3 \mid A^{-1})$$

Indeed

$$\boxed{A^{-1} = E_{10} \cdots E_1 = \frac{1}{2}\begin{pmatrix} 2 & -3 & 3 \\ -3 & 4 & -3 \\ 3 & -3 & 2 \end{pmatrix}}$$

as can be verified:

$$A^{-1}A = \frac{1}{2}\begin{pmatrix} 2 & -3 & 3 \\ -3 & 4 & -3 \\ 3 & -3 & 2 \end{pmatrix}\begin{pmatrix} 1 & 3 & 3 \\ 3 & 5 & 3 \\ 3 & 3 & 1 \end{pmatrix}$$

$$= \frac{1}{2}\begin{pmatrix} 2-\cancel{9}+\cancel{9} & \cancel{6}-\cancel{15}+\cancel{9} & \cancel{6}-\cancel{9}+\cancel{3} \\ \cancel{-3}+\cancel{12}-\cancel{9} & -9+20-9 & \cancel{-9}+\cancel{12}-\cancel{3} \\ \cancel{3}-\cancel{9}+\cancel{6} & \cancel{9}-\cancel{15}+\cancel{6} & \cancel{9}-\cancel{9}+2 \end{pmatrix} = \frac{1}{2}\begin{pmatrix} 2 & 0 & 0 \\ 0 & 2 & 0 \\ 0 & 0 & 2 \end{pmatrix}$$

$$= \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix} = I_3 \quad \boxtimes$$

pf: If $B = E_k \cdots E_1$, for elementary matrices $E_1, \dots, E_k \in \mathcal{E}_n \subseteq GL(n,\mathbb{R})$, then $B \in GL(n,\mathbb{R})$ & $B^{-1} = (E_k \cdots E_1)^{-1} = E_1^{-1} \cdots E_k^{-1}$, so if:

$$\boxed{B(A \,|\, I_n) = (I_n \,|\, B)}$$

then

$$\boxed{BA = I_n}$$

since

$$
\begin{aligned}
(I_n \,|\, B) &\overset{\text{by assump.}}{=} B(A \,|\, I_n) \\
&= B\left(\vec{a}_1 \cdots \vec{a}_n \,\middle|\, \vec{e}_1 \cdots \vec{e}_n\right) \\
&\overset{\text{def}}{=} \left(B\vec{a}_1 \cdots B\vec{a}_n \,\middle|\, B\vec{e}_1 \cdots B\vec{e}_n\right) \\
&= (BA \,|\, B)
\end{aligned}
$$

But this means $B = A^{-1}$; for

$$
\begin{aligned}
BA = I_n \implies A &= (B^{-1}B)A \\
&= B^{-1}(BA) \\
&= B^{-1} I_n = B^{-1}
\end{aligned}
$$

$$\implies A^{-1} = (B^{-1})^{-1} = B.$$

Conversely, if $A \in GL(n,\mathbb{R})$, then observe that

$$A^{-1}(A \mid I_n) = A^{-1}(\vec{a}_1 \cdots \vec{a}_n \mid \vec{e}_1 \cdots \vec{e}_n)$$

$$= (A^{-1}\vec{a}_1 \cdots A^{-1}\vec{a}_n \mid A^{-1}\vec{e}_1 \cdots A^{-1}\vec{e}_n)$$

$$\left(\begin{array}{l} A^{-1}\vec{a}_i = \vec{e}_i \\ \iff A^{-1}A = I_n \end{array} \right) \qquad = (\vec{e}_1 \cdots \vec{e}_n \mid \vec{a}_1^{-1} \cdots \vec{a}_n^{-1})$$

in terms of columns

$$= (I_n \mid A^{-1})$$

On the other hand $A \in GL(n,\mathbb{R}) \iff N(A) = \{\vec{0}\}$

$$\iff \operatorname{rank} A = \dim \mathbb{R}^n - \operatorname{null} A$$

$$= n - 0 = n$$

because $\operatorname{rank} A$
$= \dim(R(A))$
$= \dim(R(\operatorname{rref} A))$
$= \operatorname{rank}(\operatorname{rref} A)$

bc. $\operatorname{rref} A$ is gotten by left-mult. by $E_i \in \mathcal{E}_n \subseteq GL(n,\mathbb{R})$ & E_i preserve dimensions/bases.

$$\iff \operatorname{rank} A = \operatorname{rank}(\operatorname{rref} A) = n$$

$$\iff \operatorname{null}(\operatorname{rref} A) = 0$$

$$\iff \operatorname{rref} A = I_n$$

$$\iff A^{-1} = E_k \cdots E_1$$

for some $E_i \in \mathcal{E}_n$.

QED

Thm. 2 For any $A \in M_n(F)$, the <u>characteristic polynomial</u>

$$\boxed{c_A(\lambda) \stackrel{\text{def}}{=} \det(A - \lambda I)}$$

has degree n,

$$\boxed{c_A(\lambda) \in \mathbb{P}_n}$$

and therefore has at most n roots, but possibly fewer if $F = \mathbb{R}$ (it will have <u>exactly</u> n roots over $F = \mathbb{C}$).

Pf: Tedious, postponed.

Thm. 3 For any operator $T \in \mathcal{L}(V)$ on a real or complex vector space of finite dimension n, and for any basis β for V,

$$\boxed{\begin{array}{c} \lambda \in F \text{ is an eigenvalue of } T \in \mathcal{L}(V) \\ \text{w/ corresponding eigenvector } \vec{v} \in V, \\ \boxed{T(\vec{v}) = \lambda \vec{v}} \\ \Longleftrightarrow \\ \lambda \text{ is an eigenvalue of } [T]_\beta \in M_n(F) \\ \text{w/ corresponding eigenvector } [\vec{v}]_\beta \in F^n \\ \boxed{[T]_\beta [\vec{v}]_\beta = \lambda [\vec{v}]_\beta} \end{array}}$$

pf: $T(\vec{v}) = \lambda \vec{v} \implies \lambda [\vec{v}]_\beta = \lambda \varphi_\beta(\vec{v}) = \varphi_\beta(\lambda \vec{v})$

$$= \varphi_\beta(T(\vec{v})) = [T(\vec{v})]_\beta$$

$$= [T]_\beta [\vec{v}]_\beta$$

Conversely, $[T]_\beta [\vec{v}]_\beta = \lambda [\vec{v}]_\beta$, then

$$\begin{aligned}
\varphi_\beta(\lambda \vec{v}) &= \lambda \varphi_\beta(\vec{v}) \\
&= \lambda [\vec{v}]_\beta \\
&= [T]_\beta [\vec{v}]_\beta \quad \text{(by assump.)} \\
&= [T(\vec{v})]_\beta \quad \text{(Thm.)} \\
&= \varphi_\beta(T(\vec{v}))
\end{aligned}$$

But since $\varphi_\beta \in \mathrm{Isom}(V, F^n)$, we can invert it,

$$\begin{aligned}
\lambda \vec{v} &= \varphi_\beta^{-1}(\varphi_\beta(\lambda \vec{v})) \\
&= \varphi_\beta^{-1}(\varphi_\beta(T(\vec{v}))) \\
&= T(\vec{v})
\end{aligned}$$

QED

Observations: (1) If $T \in \mathcal{L}(V)$ has n distinct eigenvalues $\lambda_1, \ldots, \lambda_n \in F$, then T is diagonalizable.
($\dim V = n$)

(by (4) of "Eigenvalues I")

(2) If $T \in \mathcal{L}(V)$ is diagonalizable, then the characteristic polynomial of T splits/factors completely,

$$\boxed{\begin{aligned} c_T(\lambda) &\overset{\text{def}}{:=} \det(T - \lambda I) \\ &= (\lambda - \lambda_1)(\lambda - \lambda_2)\cdots(\lambda - \lambda_n) \end{aligned}}$$

Here the λ_i's are not assumed distinct.

pf: Exercise. Hint: see "Eigenvalues I," Prop. 1.

<u>Thm. 4</u> If $\lambda \in F = \mathbb{R}$ or $\mathbb{C}$ is an eigenvalue of $T \in \mathcal{L}(V)$, then the corresponding set of λ-eigenvectors

$$\boxed{E_\lambda \overset{\text{def}}{=} \{\vec{v} \in V \,/\, T(\vec{v}) = \lambda \vec{v}\}}$$

is a <u>subspace of V</u>, called the <u>λ-eigenspace</u>.

<u>Important Remark:</u>

$$\boxed{\begin{aligned} E_\lambda &= \{\vec{v} \in V \,/\, (T - \lambda I)\vec{v} = \vec{0}\} \\ &= N(T - \lambda I) \end{aligned}}$$

<u>Pf:</u> E_λ is nonempty by assumption, since λ is an eigenvalue, so must correspond to some (necessarily nonzero) eigenvector $\vec{v} \in V$, i.e. $\vec{v} \in E_\lambda$ for at least one such $\vec{v}$.

(1) If $\vec{u}, \vec{v} \in E_\lambda$, then $T(\vec{u}) = \lambda \vec{u}$ & $T(\vec{v}) = \lambda \vec{v}$,

so
$$\begin{aligned} T(\vec{u} + \vec{v}) &= T(\vec{u}) + T(\vec{v}) \quad (T \text{ is } \underline{\text{linear}}!) \\ &= \lambda \vec{u} + \lambda \vec{v} \\ &= \lambda(\vec{u} + \vec{v}) \end{aligned}$$

$$\Longrightarrow \vec{u} + \vec{v} \in E_\lambda$$

(2) $\vec{v} \in E_\lambda$ and $c \in F \implies T(\vec{v}) = \lambda \vec{v}$

$\implies T(c\vec{v}) = c\,T(\vec{v})$

(T is linear)

$= c\lambda\vec{v}$

$= \lambda(c\vec{v})$

$\implies c\vec{v} \in E_\lambda$.

Thm. 5 For any eigenvalue λ_0 of $T \in \mathcal{L}(V)$, consider the two numbers,

$$\boxed{\begin{aligned} k &= \underline{\text{algebraic multiplicity of } \lambda_0} \\ &\stackrel{\text{def}}{=} \text{highest power } k \text{ of } (\lambda-\lambda_0) \text{ in} \\ &\quad c_T(\lambda) = \det(T-\lambda I) \end{aligned}}$$

$$\boxed{\begin{aligned} l &= \underline{\text{geometric multiplicity of } \lambda_0} \\ &\stackrel{\text{def.}}{=} \boxed{\dim E_{\lambda_0} \text{ in } V} \end{aligned}}$$

$= \dim N(A - \lambda I)$ or $\dim N(T-\lambda I)$

$= \text{null}(A-\lambda I)$ or $\text{null}(T-\lambda I)$

Then, $\boxed{1 \le l = \dim E_\lambda \le k}$

pf: Choose a basis $\beta_{\lambda_0} = (\vec{b}_1, \dots, \vec{b}_l)$ for E_λ and extend it to a basis $\beta = \beta_{\lambda_0} \cup \gamma = (\vec{b}_1, \dots, \vec{b}_l, \vec{b}_{l+1}, \dots, \vec{b}_n)$ for V and use the fact that

$$T(\vec{b}_i) = \lambda_0 \vec{b}_i \text{ for all } \vec{b}_i \in \beta_{\lambda_0}$$

$$\Rightarrow [T(\vec{b}_i)]_\beta = \lambda_0 \vec{e}_i$$

Then,

$$[T]_{\beta = \beta_{\lambda_0} \cup \gamma} = \left([T]_{\beta_{\lambda_0}} \; [T]_\gamma \right)$$

$$= \begin{pmatrix} \lambda_0 I_l & B \\ 0 & C \end{pmatrix}$$

$\Longrightarrow$

$$c_T(\lambda) = \det(T - \lambda I)$$

$$= \det([T]_\beta - \lambda I_n)$$

$$= \det \begin{pmatrix} (\lambda - \lambda_0) I_l & B \\ 0 & C - \lambda I_{n-l} \end{pmatrix}$$

$$= (\lambda - \lambda_0)^l \det(C - \lambda I_{n-l})$$

Now, since $(\lambda - \lambda_0)^l$ divides $c_T(\lambda)$, & since by assumption $(\lambda - \lambda_0)^k$ divides $c_T(\lambda)$ & k is its (algebraic mult.), $1 \le l \le k$, & $l = \dim E_{\lambda_0}$. QED big "if" in the case of $\mathbb{R}$

Thm. 6 If $T \in \mathcal{L}(V)$ and the characteristic polynomial of T factors completely,

$$\boxed{c_T(\lambda) = (-1)^n(\lambda-\lambda_1)^{n_1}(\lambda-\lambda_2)^{n_2}\cdots(\lambda-\lambda_k)^{n_k}}$$

then

$$\boxed{\begin{array}{l} T \text{ is diagonalizable} \\ \qquad \iff \dim E_{\lambda_i} = n_i \\ \qquad\qquad \text{for all } i = 1, \ldots, k \end{array}}$$

key fact!

This is what you use.

Moreover, in this case, if β_i is a basis for E_{λ_i}, then

$$\boxed{\beta = \beta_1 \cup \beta_2 \cup \cdots \cup \beta_k}$$

is a basis for V consisting of eigenvectors of T, &

$$\boxed{\begin{aligned} V &= E_{\lambda_1} \oplus E_{\lambda_2} \oplus \cdots \oplus E_{\lambda_k} \\ &= \operatorname{span}\beta_1 \oplus \operatorname{span}\beta_2 \oplus \cdots \oplus \operatorname{span}\beta_k \end{aligned}}$$

Proof postponed temporarily, to allow recovery from shock of abstraction :)

so:

$$[A]_\beta = \begin{pmatrix} \lambda_1 I_{n_1} & & 0 \\ & \lambda_2 I_{n_2} & \\ 0 & & \ddots \; \lambda_k I_{n_k} \end{pmatrix}$$

Is $P \in \mathcal{L}(\mathbb{P}_3)$ diagonalizable?

ex. 2 $P = \frac{d}{dx} - 1$ on $\mathbb{P}_3$

$P(1) = -1$

$P(x) = 1 - x$

$P(x^2) = 2x - x^2$

$P(x^3) = 3x^2 - x^3$

$$[P]_\beta = \begin{pmatrix} -1 & 1 & 0 & 0 \\ 0 & -1 & 2 & 0 \\ 0 & 0 & -1 & 3 \\ 0 & 0 & 0 & -1 \end{pmatrix}$$

So $\lambda = 1 \implies$

$$[P - 1I]_\beta = \begin{pmatrix} 0 & 1 & 0 & 0 \\ 0 & 0 & 2 & 0 \\ 0 & 0 & 0 & 3 \\ 0 & 0 & 0 & 0 \end{pmatrix}$$

whose null space contains only $\vec{e}_1 = \begin{pmatrix} 1 \\ 0 \\ 0 \\ 0 \end{pmatrix} \in \mathbb{R}^4$

$\implies$ 1 is the only eigenvector of P,

$\implies \dim E_\lambda = \operatorname{span}(1)$

To be demonstrated below. $\Big\} \implies$ $\boxed{P \text{ is not diagonalizable}}$

ex.3 $A = \begin{pmatrix} 5 & -8 & 1 \\ 0 & 0 & 7 \\ 0 & 0 & -2 \end{pmatrix}$ is diagonalizable because it has 3 distinct eigenvalues, $\lambda_1 = 5$, $\lambda_2 = 0$, $\lambda_3 = -2$.

ex.4 $A = \begin{pmatrix} 1 & 3 & 3 \\ -3 & -5 & -3 \\ 3 & 3 & 1 \end{pmatrix}$

$$\Rightarrow \det(A - \lambda I_3) = -\lambda^3 - 3\lambda^2 + 4$$
$$= -(\lambda-1)(\lambda+2)^2$$
$$= 0$$

algebraic mult. of $\lambda_1 = 1$ is 1

alg. mult of $\lambda_2 = -2$ is 2

$$\Rightarrow \lambda_1 = 1$$
$$\lambda_2 = -2$$

For $\lambda_1 = 1$: $A - \lambda_1 I = \begin{pmatrix} 0 & 3 & 3 \\ -3 & -6 & -3 \\ 3 & 3 & 0 \end{pmatrix}$

has $\operatorname{rref}(A - \lambda_1 I) = \begin{pmatrix} 1 & 0 & -1 \\ 0 & 1 & 1 \\ 0 & 0 & 0 \end{pmatrix}$

2 pivots means $\operatorname{rank}(A-I) = 2$

Note! $\dim E_{\lambda_1} = \dim N(A-I) = \dim \mathbb{R}^3 - \operatorname{rank}(A-I) = 3-2 = 1$

$\Rightarrow$ let $z = t$, then $x = z = t$, $y = -z = -t$

$$\Rightarrow N(A - \lambda_1 I) = \operatorname{span}\begin{pmatrix} x \\ y \\ z \end{pmatrix} = \operatorname{span}\begin{pmatrix} 1 \\ -1 \\ 1 \end{pmatrix}$$

i.e. $\boxed{\beta_1 = \left(\begin{pmatrix} 1 \\ -1 \\ 1 \end{pmatrix}\right)}$

This equals the alg. mult of $\lambda_1 = 1$, so we're

For λ_2: $A - \lambda_2 I_3 = \begin{pmatrix} 3 & 3 & 3 \\ -3 & \blacksquare & -3 \\ 3 & 3 & 3 \end{pmatrix}$

$\Longrightarrow \operatorname{rref}(A - \lambda_2 I_3) = \begin{pmatrix} 1 & 1 & 1 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix}$

1 pivot $\Longrightarrow \operatorname{rank}(A - \lambda_2 I) = 1 \Longrightarrow \dim E_{\lambda_2} = \dim N(A - \lambda_2 I) = \dim \mathbb{R}^3 - \operatorname{rank}(A \ldots = 3 - 1 = 2$

& this = alg. mult of $\lambda_2 = -\ldots$ so we're good!

$\Longrightarrow$ let $y = s$, $z = t$, then $x = -s - t$

$\Longrightarrow \begin{pmatrix} x \\ y \\ z \end{pmatrix} = \begin{pmatrix} -s-t \\ s \\ t \end{pmatrix} = s\begin{pmatrix} -1 \\ 1 \\ 0 \end{pmatrix} + t\begin{pmatrix} -1 \\ 0 \\ 1 \end{pmatrix}$

$\Longrightarrow \boxed{\beta_2 = \left(\begin{pmatrix} -1 \\ 1 \\ 0 \end{pmatrix}, \begin{pmatrix} -1 \\ 0 \\ 1 \end{pmatrix}\right) \text{ basis for } E_{\lambda_2} = E_{-2}}$

Thus,

$$E_1 = \operatorname{span} \beta_1 = \operatorname{span}\begin{pmatrix} 1 \\ -1 \\ 1 \end{pmatrix}$$

$$E_{-2} = \operatorname{span} \beta_2 = \operatorname{span}\begin{pmatrix} -1 \\ 1 \\ 0 \end{pmatrix} \oplus \operatorname{span}\begin{pmatrix} -1 \\ 0 \\ 1 \end{pmatrix}$$

and

$$\mathbb{R}^3 = E_1 \oplus E_{-2}$$

with

$$[A]_{\beta = \beta_1 \cup \beta_2} = \begin{pmatrix} 1 & 0 & 0 \\ 0 & -2 & 0 \\ 0 & 0 & -2 \end{pmatrix}$$

ex. 5 $A = \begin{pmatrix} 2 & 4 & 3 \\ -4 & -6 & -3 \\ 3 & 3 & 1 \end{pmatrix} \in M_3(\mathbb{R})$.

Q: Is A diagonalizable? If so, find a diagonalizing basis $\delta = (d_1, \dots, d_n)$, & use it to write A as $D = P^{-1}AP$, $P = M_\delta$.

Soln: Let's compute the characteristic polynomial & find its roots, the eigenvalues of A:

① (compute $c_A(\lambda)$)

$$c_A(\lambda) = \det(A - \lambda I_3)$$

$$= \det\left(\begin{pmatrix} 2 & 4 & 3 \\ -4 & -6 & -3 \\ 3 & 3 & 1 \end{pmatrix} - \begin{pmatrix} \lambda & 0 & 0 \\ 0 & \lambda & 0 \\ 0 & 0 & \lambda \end{pmatrix}\right)$$

$$= \det\begin{pmatrix} 2-\lambda & 4 & 3 \\ -4 & -6-\lambda & -3 \\ 3 & 3 & 1-\lambda \end{pmatrix}$$

$$= (2-\lambda)\det\begin{pmatrix} -6-\lambda & -3 \\ 3 & 1-\lambda \end{pmatrix} - 4\det\begin{pmatrix} -4 & -3 \\ 3 & 1-\lambda \end{pmatrix} + 3\det\begin{pmatrix} -4 & -6-\lambda \\ 3 & 3 \end{pmatrix}$$

$$= (2-\lambda)\left[(-6-\lambda)(1-\lambda) - 3(-3)\right] - 4\left[-4(1-\lambda) - 3(-3)\right] + 3\left[-4\cdot 3 - 3(-6-\lambda)\right]$$

$$= (2-\lambda)\left(\lambda^2+6\lambda-\lambda-6+9\right)$$
$$-4\left(4\lambda-4+9\right)+3\left(-12+18+3\lambda\right)$$

1st

$$= (2-\lambda)\left(\lambda^2+5\lambda+3\right)-4\left(4\lambda+5\right)$$

2nd

$$+3\left(3\lambda+6\right)$$

$$\begin{aligned} = &-\lambda^3-5\lambda^2-3\lambda && \text{1st}\\ &+2\lambda^2+10\lambda+6 && \text{2nd}\\ &-16\lambda-20 && \text{etc.}\\ &+9\lambda+18 \end{aligned}$$

$$= -\lambda^3-3\lambda^2+4$$

② At this point you either go to Wolfram, or use your wits, to <u>find its roots</u>: let's try the latter. Since

$$c_A(0) = 4 > 0$$
$$c_A(2) = -16 < 0$$

The <u>Intermediate</u> Value Thm. tells us there is a root btw. 0 & 2. Trying the midpt. btw. 0 & 2, we are really lucky to get a root, $c_A(1) = -4+4 = 0$!

Therefore, $(\lambda-1)$ divides $-\lambda^3-3\lambda^2+4$!

We again use our wits to see how exactly it divides: either <u>long division</u> or <u>synthetic division</u> will work. I'll try to use synthetic:

coefficients of $c_A(\lambda) = -\lambda^3-3\lambda^2+4$ are $-1, -3, 0, 4$, & we know $\lambda=1$ is a root:

② add ③ add

$$\begin{array}{r|rrrr} 1 & -1 & -3 & 0 & 4 \\ & & -1 & -4 & -4 \\ \hline & \boxed{-1} & \boxed{-4} & \boxed{-4} & 0 \end{array}$$

put put

① carry it down, then mult. by 1

these are the coefficients of $-\lambda^2-4\lambda-4$

Thus, $(-\lambda^2-4\lambda-4)(\lambda-1) = -\lambda^3-4\lambda^2-\cancel{4\lambda} + \lambda^2+\cancel{4\lambda}+4$

1st 2nd

$= -\lambda^3-3\lambda^2+4$

$= c_A(\lambda)$ ✓

So indeed

$$\boxed{\begin{aligned} c_A(\lambda) &= -\lambda^3 - 3\lambda^2 + 4 \\ &= (-\lambda^2 - 4\lambda - 4)(\lambda - 1) \\ &= -\underbrace{(\lambda^2 + 4\lambda + 4)}_{\text{factors!}}(\lambda - 1) \\ &= -(\lambda + 2)^2(\lambda - 1) \end{aligned}}$$

We've factored it! Therefore, $c_A(\lambda) = 0 \Rightarrow$

$\lambda_1 = -2$ with algebraic multiplicity $= 2$

$\lambda_2 = 1$ alg. mult. $= 1$

③ Let us now find the geometric multiplicities

$$\boxed{\begin{aligned} \dim E_{\lambda_1} &= \dim E_{-2} \\ \dim E_{\lambda_2} &= \dim E_1 \end{aligned}}$$

For this, as we know, we need eigenvectors:

Let's start with $\lambda_1 = -2$:

$$A + 2I = \begin{pmatrix} 2+2 & 4 & 3 \\ -4 & -6+2 & -3 \\ 3 & 3 & 1+2 \end{pmatrix}$$

$$= \begin{pmatrix} 4 & 4 & 3 \\ -4 & -4 & -3 \\ 3 & 3 & 3 \end{pmatrix}$$

Row-reduce it to find the eigenvectors $\vec{v} \in N(A+2A)$

$$\begin{pmatrix} 4 & 4 & 3 \\ -4 & -4 & -3 \\ 3 & 3 & 3 \end{pmatrix} \longrightarrow \begin{pmatrix} 1 & 1 & 3/4 \\ 0 & 0 & 0 \\ 1 & 1 & 1 \end{pmatrix}.$$

$$\longrightarrow \begin{pmatrix} 1 & 1 & 3/4 \\ 0 & 0 & 0 \\ 0 & 0 & 1/4 \end{pmatrix}$$

$$\longrightarrow \begin{pmatrix} 1 & 1 & 0 \\ 0 & 0 & 1 \\ 0 & 0 & 0 \end{pmatrix}$$

2 pivots $\Longrightarrow$ rank$(A+2I) = 2$

$$\Longrightarrow \boxed{\text{null}(A+2I) \stackrel{\text{def}}{=} \dim N(A+2I) \overset{\text{Rank-Nullity}}{=} \dim \mathbb{R}^3 - \text{rank}(A+2I) = 3-2 = 1}$$

remember,

$E_{-2} = N(A-(-2)I)$

$= N(A+2I)$, so $\dim E_{-2} = \dim N(A+2I) = 1$

Thus,

$$\begin{aligned}
&\text{geometric multiplicity of } \lambda_1 = -2 \\
&\quad = \dim E_{-2} \\
&\quad = \dim (N(A+2I)) \\
&\quad = 1 \\
&\quad < 2 \\
&\quad = \text{algebraic multiplicity of } \lambda_1 = -2
\end{aligned}$$

so by Thm. 6 **A is not diagonalizable.**

We're done, there's nothing to be done.

But for the record,

$$\begin{aligned}
N(A - \lambda_2 I) &= N(\mathrm{rref}(A - \lambda_2 I)) \\
&= N\left(\mathrm{rref}\begin{pmatrix} 1 & 4 & 3 \\ -4 & -7 & -3 \\ 3 & 3 & 0 \end{pmatrix}\right) \\
&= N\begin{pmatrix} 1 & 0 & -1 \\ 0 & 1 & 1 \\ 0 & 0 & 0 \end{pmatrix} = \mathrm{span}\begin{pmatrix} 1 \\ -1 \\ 1 \end{pmatrix}
\end{aligned}$$

$\vec{v}_2 \in E_1$

$= E_{\lambda_2}$

$= \mathrm{span}(\vec{v}_2)$

2 pivots $\Rightarrow$

rank $= 2 \Rightarrow$ null $= 1 =$ alg. mult 1

$\boxtimes$

ex. 6 Let $T \in \mathcal{L}(\mathbb{P}_2)$ be defined by

$$\boxed{T(p(x)) \stackrel{\text{def}}{=} p(1) + p'(0)x + \big(p'(0) + p''(0)\big)x^2}$$

or if you prefer,

$$T = e_1 + x\frac{d}{dx}\Big|_{x=0} + x^2\left(\frac{d}{dx}\Big|_{x=0} + \frac{d^2}{dx^2}\Big|_{x=0}\right)$$

$$= e_1 + (x+x^2)\frac{d}{dx}\Big|_{x=0} + x^2\frac{d^2}{dx^2}\Big|_{x=0}$$

evaluation-at-$x=1$ function ("plug in 1")

It's easier to see that T is linear this way. In general, a differential operator is linear if it is of the form $T = e_{a_1} + \cdots + e_{a_k} + f_1\frac{d}{dx} + f_2\frac{d^2}{dx^2} + \cdots + f_r\frac{d^r}{dx^r}$, since all derivatives $\frac{d^i}{dx^i}$ are linear.

Anyway, let's find eigenvalues & determine whether T is diagonalizable or not, & diagonalize it if if it. Let's take the standard basis in $\mathbb{P}_2$,

$$\sigma = (1, x, x^2)$$

and represent T in σ as a matrix,

$$[T]_\sigma \stackrel{\text{def}}{=} \Big([T(1)]_\sigma \;\; [T(x)]_\sigma \;\; [T(x^2)]_\sigma \Big)$$

Well, let's compute those columns:

(i) $$T(1) = e_1(1) + (x+x^2)\underbrace{\frac{d}{dx}\Big|_{x=0}(1)}_{=0} + x^2\underbrace{\frac{d^2}{dx^2}\Big|_{x=0}(1)}_{=0}$$

$$= 1 = \boxed{1}\cdot 1 + \boxed{0}\cdot x + \boxed{0}\cdot x^2 \Longrightarrow [T(1)]_\sigma = \begin{pmatrix}1\\0\\0\end{pmatrix}$$

(ii) $$T(x) = e_1(x) + (x+x^2)\underbrace{\frac{d}{dx}\Big|_{x=0}(x)}_{=1} + x^2\underbrace{\frac{d^2}{dx^2}\Big|_{x=0}(x^2)}_{=0}$$

$$= 1 + (x+x^2)\cdot 1 = \boxed{1}\cdot 1 + \boxed{1}\cdot x + \boxed{1}\cdot x^2$$

$$\Longrightarrow [T(x)]_\sigma = \begin{pmatrix}1\\1\\1\end{pmatrix}$$

(iii) $$T(x^2) = e_1(x^2) + (x+x^2)\underbrace{\frac{d}{dx}\Big|_{x=0}(x^2)}_{=0} + x^2\underbrace{\frac{d^2}{dx^2}\Big|_{x=0}(x^2)}_{=2}$$

$$= 1 + 2x^2 = \boxed{1}\cdot 1 + \boxed{0}\cdot x + \boxed{2}\cdot x^2 \Longrightarrow [T(x^2)] = \begin{pmatrix}1\\0\\2\end{pmatrix}$$

Therefore,

$$[T]_\sigma = \begin{pmatrix} 1 & 1 & 1 \\ 0 & 1 & 0 \\ 0 & 1 & 2 \end{pmatrix}$$

so since λ is an eigenvalue of $T \iff \lambda$ is an eigenvalue of $[T]_\sigma$ (Thm. 3, p. 8), we compute

$$\begin{aligned}
c_T(\lambda) &= c_{[T]_\sigma}(\lambda) \\
&= \det([T]_\sigma - \lambda I_3) \\
&= \det\left(\begin{pmatrix} 1 & 1 & 1 \\ 0 & 1 & 0 \\ 0 & 1 & 2 \end{pmatrix} - \begin{pmatrix} \lambda & 0 & 0 \\ 0 & \lambda & 0 \\ 0 & 0 & \lambda \end{pmatrix}\right) \\
&= \det\begin{pmatrix} 1-\lambda & 1 & 1 \\ 0 & 1-\lambda & 0 \\ 0 & 1 & 2-\lambda \end{pmatrix} \\
&= (1-\lambda)\det\begin{pmatrix} 1-\lambda & 0 \\ 1 & 2-\lambda \end{pmatrix} - 1 \cdot \underbrace{\det\begin{pmatrix} 0 & 0 \\ 0 & 2-\lambda \end{pmatrix}}_{=0} \\
&\qquad + 1 \cdot \underbrace{\det\begin{pmatrix} 0 & 1-\lambda \\ 0 & 1 \end{pmatrix}}_{=0} \\
&= (1-\lambda)((1-\lambda)(2-\lambda) - 0) = (1-\lambda)^2(2-\lambda)
\end{aligned}$$

$\lambda_1 = 1$, alg. mult 2

$\lambda_2 = 2$, alg. mult = 1

Now we're gotta compute geometric multiplicities,

$$
\begin{aligned}
\dim E_{\lambda_1} &= \dim E_1 \\
&= \dim N(T - I) \\
&= \dim N([T]_\sigma - I_3) \\
&= \dim N\begin{pmatrix} 0 & 1 & 1 \\ 0 & 0 & 0 \\ 0 & 1 & 1 \end{pmatrix} \\
&= \dim N\begin{pmatrix} 0 & 1 & 1 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix} \leftarrow \text{has rank } 1 \\
&= \dim \mathbb{R}^3 - \operatorname{rank}\begin{pmatrix} 0 & 1 & 1 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix} \\
&= 3 - 1 \\
&= 2 \leftarrow \text{this is good, it equals the algebraic mult.!}
\end{aligned}
$$

and

$$
\begin{aligned}
\dim E_{\lambda_2} &= \dim E_2 \\
&= \dim N(T - 2I) \\
&= \dim N([T]_\sigma - 2I_3)
\end{aligned}
$$

$$= \dim N\begin{pmatrix} -1 & 1 & 1 \\ 0 & -1 & 0 \\ 0 & 1 & 0 \end{pmatrix}$$ ← these null spaces are equal since

$$= \dim N\begin{pmatrix} 1 & 0 & -1 \\ 0 & 1 & 0 \\ 0 & 0 & 0 \end{pmatrix}$$

$\sim_S = \sim_R$

↑ col'n equiv. ↑ row-equiv.

2 pivots $\Rightarrow$ rank $= 2$

$\Rightarrow$ null $= 1$ $(= 3-2)$

$= 1$ ← also good, it equals the alg. mult. of $\lambda_2 = 2$!

Good! T is diagonalizable, by dimension count. All that's left is to find a diagonalizing basis, which means find bases for $E_{\lambda_1} = N(T-I)$ & $E_{\lambda_2} = N(T-2I)$.

It's easier to find $[\vec{v}_i]_\sigma$ & get $\vec{v}_i$ as $\varphi_\sigma^{-1}([\vec{v}_i]_\sigma)$, of course, so this is what we do: bec. $\varphi_\sigma(1) = \vec{e}_1$, $\varphi_\sigma(x) = \vec{e}_2$, $\varphi_\sigma(x^2) = \vec{e}_3$

Well,

$$N([T]_\sigma - I_3) = N\begin{pmatrix} 0 & 1 & 1 \\ 0 & 0 & 0 \\ 0 & 1 & 1 \end{pmatrix}$$

$$= N\begin{pmatrix} 0 & 1 & 1 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix}$$

$$\Longrightarrow \quad x = s \text{ for any } s \in \mathbb{R}$$
$$z = t, \quad t \in \mathbb{R}$$
$$y = -z = -t$$

$$\Longrightarrow \quad \vec{x} \in N\begin{pmatrix} 0 & 1 & 1 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix} \Longleftrightarrow$$

$$\vec{x} = \begin{pmatrix} x \\ y \\ z \end{pmatrix} = \begin{pmatrix} s \\ -t \\ t \end{pmatrix}$$

$$= s\underbrace{\begin{pmatrix} 1 \\ 0 \\ 0 \end{pmatrix}}_{=[\vec{v}_1]_\sigma} + t\underbrace{\begin{pmatrix} 0 \\ -1 \\ 1 \end{pmatrix}}_{=[\vec{v}_2]_\sigma}$$

a polynomial

Since $[\vec{v}_1]_\sigma = \begin{pmatrix} 1 \\ 0 \\ 0 \end{pmatrix}$, $\boxed{\begin{aligned}\vec{v}_1 &= 1\cdot 1 + 0\cdot x + 0\cdot x^2 \\ &= 1\end{aligned}}$

& $[\vec{v}_2]_\sigma = \begin{pmatrix} 0 \\ -1 \\ 1 \end{pmatrix} \Longrightarrow \boxed{\begin{aligned}\vec{v}_2 &= 0\cdot 1 + (-1)x + 1\cdot x^2 \\ &= -x + x^2\end{aligned}}$

$\Big\} \in E_{\lambda_1} = E_1$

another polynomial

Thus,

$$\boxed{\begin{array}{l} \vec{v}_1 = 1,\ \vec{v}_2 = -x + x_2 \in E_{\lambda_1} = N(T - I), \\ \quad \text{form a basis for } E_{\lambda_1}, \\ \quad \beta_1 = (1, -x+x^2) \end{array}}$$

Indeed,

$$T(1) = 1 = 1 \cdot 1 \quad \checkmark$$

$$\begin{aligned} T(-x+x^2) &= -T(x) + T(x^2) \\ &= -(1+x+x^2) + (1+2x^2) \\ &= -x + x^2 \\ &= 1 \cdot (-x+x^2) \quad \checkmark \end{aligned}$$

(see p 25)

Now let's do the other one:

$$N([T]_\sigma - 2I_3) = N\begin{pmatrix} -1 & 1 & 1 \\ 0 & -1 & 0 \\ 0 & 1 & 0 \end{pmatrix}$$

$$= N\begin{pmatrix} 1 & 0 & -1 \\ 0 & 1 & 0 \\ 0 & 0 & 0 \end{pmatrix}$$

$$\begin{array}{ccc} \uparrow & \uparrow & \uparrow \\ \text{pivots} & & \text{free} \end{array}$$

$\Longrightarrow$ let $z = t$, $\quad x = z = t$, too

$y = 0$

so $\vec{x} = \begin{pmatrix} x \\ y \\ z \end{pmatrix} \in N([T]_\sigma - 2I_3)$ iff

$$\vec{x} = \begin{pmatrix} t \\ 0 \\ t \end{pmatrix} = t\begin{pmatrix} 1 \\ 0 \\ 1 \end{pmatrix}$$

basis vector rep.

$[\vec{v}_3]_\sigma$, $\vec{v}_3 \in E_{\lambda_2} = N(T-\lambda_2 I)$

Therefore,

$$[\vec{v}_3]_\sigma = \begin{pmatrix} 1 \\ 0 \\ 1 \end{pmatrix} \Longrightarrow \boxed{\begin{aligned} \vec{v}_3 &= 1\cdot 1 + 0\cdot x + 1\cdot x^2 \\ &= 1+x^2 \in E_{\lambda_2} = N(T-2I) \end{aligned}}$$

Indeed,

$$\begin{aligned} T(1+x^2) &= T(1) + T(x^2) \\ &= 1 + (1+2x^2) \\ &= 2+2x^2 \\ &= 2(1+x^2) \quad \checkmark \end{aligned}$$

Let $\boxed{\beta_2 = (1+x^2), \text{ basis for } E_{\lambda_2} = E_2}$

and note,

$$\boxed{\beta = \beta_1 \cup \beta_2 = (1, -x+x^2, 1+x^2)}$$

is a diagonalizing basis for $T = e_1 + (x+x^2)\frac{d}{dx}\Big|_{x=0} + x^2\frac{d^2}{dx^2}\Big|_{x=0} \in \mathcal{L}(\mathbb{P}_2)$

and

$$\boxed{[T]_\beta = \Big([T(1)]_\beta \;\; [T(-x+x^2)]_\beta \;\; [T(1+x^2)]_\beta \Big) \overset{\text{p. 30-31}}{=} \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 2 \end{pmatrix} = D}$$

while

$$\boxed{P = M_{\beta\sigma} = \Big([1]_\sigma \;\; [-x+x^2]_\sigma \;\; [1+x^2]_\sigma \Big) = \begin{pmatrix} 1 & 0 & 1 \\ 0 & -1 & 0 \\ 0 & 1 & 1 \end{pmatrix}}$$

$\rightsquigarrow$ det $= -1 \neq 0$, so $P \in GL(3,\mathbb{R})$ ✓

satisfies

$$PDP^{-1} = \begin{pmatrix} 1 & 0 & 1 \\ 0 & -1 & 0 \\ 0 & 1 & 1 \end{pmatrix}\begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 2 \end{pmatrix}\underbrace{\begin{pmatrix} 1 & -1 & -1 \\ 0 & -1 & 0 \\ 0 & 1 & 1 \end{pmatrix}}_{\text{used Wolfram}}$$

$$= \begin{pmatrix} 1 & 0 & 2 \\ 0 & -1 & 0 \\ 0 & 1 & 2 \end{pmatrix}\begin{pmatrix} 1 & -1 & -1 \\ 0 & -1 & 0 \\ 0 & 1 & 1 \end{pmatrix} = \begin{pmatrix} 1 & 1 & 1 \\ 0 & 1 & 0 \\ 0 & 1 & 2 \end{pmatrix} = [T]_\sigma$$

✓ see top of p. 26

Lastly,

$$\beta = \beta_1 \cup \beta_2 \implies \boxed{\begin{aligned} \mathbb{P}_2 &= \operatorname{span}\beta_2 \oplus \operatorname{span}\beta_1 \\ &= \operatorname{span}(1, -x+x^2) \\ &\qquad \oplus \operatorname{span}(1+x^2) \\ &= E_1 \oplus E_2 \end{aligned}}$$

and T acts as

$$\boxed{T = \overset{=\lambda_1}{1}\cdot\underbrace{\pi_{E_1}}_{\text{projection onto } E_1} + \overset{=\lambda_2}{2}\underbrace{\pi_{E_2}}_{\text{projection onto } E_2}}$$

$$T \in \mathcal{L}(\mathbb{P}_2) \iff T = \pi_{E_1} + \pi_{E_2}$$

$$\in \mathcal{L}(E_1 \oplus E_2)$$