Eigenvalues & Eigenvectors I

(1) Let $\boxed{\mathcal{U}_n = \text{all upper triangular matrices } A \in M_n(\mathbb{R}), \text{ i.e. } A = \begin{pmatrix} a_{11} & a_{12} & \cdots & a_{1n} \\ 0 & a_{22} & \cdots & a_{2n} \\ \vdots & & \ddots & \vdots \\ 0 & \cdots & 0 & a_{nn} \end{pmatrix}, \quad a_{ij} = 0 \text{ if } i > j}$

Then $\boxed{\det A = \prod_{i=1}^{n} a_{ii}}$ and same for $A \in \underline{\mathcal{L}_n}$ (lower triangular.

<u>ex. 1.</u> $\det\begin{pmatrix} 5 & -2 & 3 \\ 0 & -4 & 1 \\ 0 & 0 & 3 \end{pmatrix} = 5 \cdot (-4) \cdot 3 = -60$

(2) Let $\boxed{\mathcal{E}_n = \text{all elementary matrices in } GL(n, \mathbb{R}).}$

and let us <u>decompose $\mathcal{E}_n$ into its 3 types</u>:

$$\boxed{\mathcal{E}_n = \mathcal{E}_n^{I} \cup \mathcal{E}_n^{II} \cup \mathcal{E}_n^{III}}$$

where $\boxed{\begin{array}{l} \mathcal{E}_n^{I} = \text{type I, \underline{row-swapping} elementary matrices} \\ \mathcal{E}_n^{II} = \text{type II, \underline{scaling a row by } } \underline{c \neq 0}\text{, elem. matrices} \\ \mathcal{E}_n^{III} = \text{type III, \underline{adding a multiple of a row to another row}, elementary matrices} \end{array}}$

Then,

(i) for any $E \in \mathcal{E}_n^{I}$ & any $A \in M_n(\mathbb{R})$

$$\boxed{\det(EA) = -\det A}$$

(ii) for any $E \in \mathcal{E}_n^{II}$, if E scales a row by $c \neq 0$,

$$\boxed{\det(EA) = c \det A}$$

(iii) $E \in \mathcal{E}_n^{III} \Longrightarrow \boxed{\det(EA) = \det A}$

ex. 2. Let $A = \begin{pmatrix} 1 & -4 & 2 \\ -2 & 8 & -9 \\ -1 & 7 & 0 \end{pmatrix} \in M_3(\mathbb{R})$.

Let us reduce A to an upper triangular matrix & apply (1) & (2):

$$A = \begin{pmatrix} 1 & -4 & 2 \\ -2 & 8 & -9 \\ -1 & 7 & 0 \end{pmatrix} \xrightarrow[E_2 \in \mathcal{E}_3^{III}]{E_1 \in \mathcal{E}_3^{III}} \begin{pmatrix} 1 & -4 & 2 \\ 0 & 0 & -5 \\ 0 & 3 & 2 \end{pmatrix}$$

$$\xrightarrow{E_3 \in \mathcal{E}_3^{I}} \begin{pmatrix} 1 & -4 & 2 \\ 0 & 3 & 2 \\ 0 & 0 & 5 \end{pmatrix} \in \mathcal{U}_3$$

where

$$E_1 = \begin{pmatrix} 1 & 0 & 0 \\ 2 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix} \quad \text{add } 2I + II$$

$$E_2 = \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 1 & 0 & 1 \end{pmatrix} \quad \text{add } I \text{ to } II$$

$$E_3 = \begin{pmatrix} 1 & 0 & 0 \\ 0 & 0 & 1 \\ 0 & 1 & 0 \end{pmatrix} \quad \text{swap } II \,\&\, III$$

Indeed,

$$E_3E_2E_1A = \underbrace{\begin{pmatrix}1&0&0\\0&0&1\\0&1&0\end{pmatrix}\begin{pmatrix}1&0&0\\0&1&0\\1&0&1\end{pmatrix}}_{\text{1st}}\underbrace{\begin{pmatrix}1&0&0\\2&1&0\\0&0&1\end{pmatrix}\begin{pmatrix}1&-4&2\\-2&8&-9\\-1&7&0\end{pmatrix}}_{\text{2nd}}$$

$$= \begin{pmatrix}1&0&0\\1&0&1\\0&1&0\end{pmatrix}\begin{pmatrix}1&-4&2\\0&0&-5\\-1&7&0\end{pmatrix}$$

$$= \begin{pmatrix}1&-4&2\\0&3&2\\0&0&-5\end{pmatrix}$$

as before. But now:

$$\det(E_3E_2E_1A) \overset{(1)}{=} 1\cdot 3\cdot(-5) = -15$$

and

$$\boxed{\begin{aligned} -15 &\overset{(1)}{=} \det(E_3E_2E_1A) \\ &\overset{\text{Thm.}}{=} (\det E_3)(\det E_2)(\det E_1)(\det A) \\ &\overset{(2)}{=} (-1)\cdot 1\cdot 1\cdot \det A \\ &= -\det A \end{aligned}}$$

$$\Longrightarrow \boxed{\det A = 15}$$

Indeed, let us verify this by using the def.:

$$\det A = \det\begin{pmatrix} 1 & -4 & 2 \\ -2 & 8 & -9 \\ -1 & 7 & 0 \end{pmatrix}$$

$$= 1\cdot\det\begin{pmatrix} 8 & -9 \\ 7 & 0 \end{pmatrix} - (-4)\det\begin{pmatrix} -2 & -9 \\ -1 & 0 \end{pmatrix} + 2\cdot\det\begin{pmatrix} -2 & 8 \\ -1 & 7 \end{pmatrix}$$

$$= 63 + 4\cdot(-9) + 2\cdot(-14+8)$$

$$= 63 - 36 - 12$$

$$= 63 - 48$$

$$= 15 \quad \checkmark$$

ex. 3 $\quad A = \begin{pmatrix} 2 & 0 & 0 & 1 \\ 0 & 1 & 3 & -3 \\ -2 & -3 & -5 & 2 \\ 4 & -4 & 4 & -6 \end{pmatrix}$ in $M_4(\mathbb{R})$.

$$\xrightarrow{E_1, E_2 \in \mathcal{E}_4^{III}} \begin{pmatrix} 2 & 0 & 0 & 1 \\ 0 & 1 & 3 & -3 \\ 0 & 0 & 4 & 3 \\ 0 & 0 & 16 & -8 \end{pmatrix} \qquad \begin{array}{l} E_1: \text{add } I \text{ to } III \\ E_2: \text{add } -2I \text{ to } IV \end{array}$$

$$\xrightarrow{E_3, E_4 \in \mathcal{E}_4^{III}} \begin{pmatrix} 2 & 0 & 0 & 1 \\ 0 & 1 & 3 & -3 \\ 0 & 0 & 4 & -6 \\ 0 & 0 & 16 & -20 \end{pmatrix} \qquad \begin{array}{l} E_3: \text{add } 3II \text{ to } III \\ E_4: \text{add } 4II \text{ to } IV \end{array}$$

$$\xrightarrow{E_5 \in \mathcal{E}_4^{III}} \begin{pmatrix} 2 & 0 & 0 & 1 \\ 0 & 1 & 3 & -3 \\ 0 & 0 & 4 & -7 \\ 0 & 0 & 0 & 4 \end{pmatrix} \qquad E_5: \text{add } -4III \text{ to } IV$$

Since $E_1, \dots, E_5 \in \mathcal{E}_4^{III}$,

$$\det A \overset{(2)}{=} \det(E_5 E_4 E_3 E_2 E_1 A)$$

$$= \det \begin{pmatrix} 2 & 0 & 0 & 1 \\ 0 & 1 & 3 & -3 \\ 0 & 0 & 4 & -7 \\ 0 & 0 & 0 & 4 \end{pmatrix}$$

$$\overset{(1)}{=} 2 \cdot 1 \cdot 4 \cdot 4$$

$$= 32$$

(3) For any $A = \begin{pmatrix} a_{11} & a_{12} & \cdots & a_{1n} \\ 0 & a_{22} & \cdots & a_{2n} \\ \vdots & & \ddots & \vdots \\ 0 & \cdots & 0 & a_{nn} \end{pmatrix} \in \mathcal{U}_n$

the eigenvalues of A are

$$\boxed{\begin{aligned} \lambda_1 &= a_{11} \\ \lambda_2 &= a_{22} \\ &\vdots \\ \lambda_n &= a_{nn} \end{aligned}}$$

counting duplicates

pf: $\det(A - \lambda I) = \det \begin{pmatrix} a_{11}-\lambda & a_{12} & \cdots & a_{1n} \\ 0 & a_{22}-\lambda & \cdots & a_{2n} \\ \vdots & & \ddots & \vdots \\ 0 & \cdots & 0 & a_{nn}-\lambda \end{pmatrix}$

$\overset{(1)}{=} (a_{11}-\lambda)(a_{22}-\lambda)\cdots(a_{nn}-\lambda)$

$= 0$

$\Longrightarrow \lambda = a_{11}, a_{22}, \cdots a_{nn}.$ QED

Similarly, for any $A = \begin{pmatrix} a_{11} & \cdots & 0 \\ \vdots & \ddots & \vdots \\ a_{n1} & \cdots & a_{nn} \end{pmatrix} \in \mathcal{L}_n$,

$\lambda_1 = a_{11}, \cdots, \lambda_n = a_{nn}$ counting duplicates ex. 4 $\quad A = \begin{pmatrix} 3 & 6 & -8 \\ 0 & 0 & 6 \\ 0 & 0 & 2 \end{pmatrix} \Longrightarrow \begin{array}{l} \lambda_1 = 3 \\ \lambda_2 = 0 \\ \lambda_3 = 2 \end{array} \Longrightarrow A \notin GL(3,\mathbb{R})$

ex. 5 $\quad A = \begin{pmatrix} 4 & 0 & 0 \\ -2 & 1 & 0 \\ 5 & 3 & 4 \end{pmatrix} \Longrightarrow \begin{array}{l} \lambda_1 = 4 \\ \lambda_2 = 1 \\ \lambda_3 = 4 \end{array} \Longrightarrow A \in GL(3,\mathbb{R})$

(4) For any $A \in M_n(F)$, <u>if $\lambda_1, \dots, \lambda_k$ are distinct eigenvalues</u> <u>then corresponding eigenvectors $\vec{v}_1, \dots, \vec{v}_k \in F^n$</u> <u>are linearly independent.</u>

pf: If not, if the $\vec{v}_i$ are linearly dependent, then let $\vec{v}_p$ be the last vector, with index $p \le k$ largest, such that $\vec{v}_p \in \operatorname{span}(\vec{v}_1, \dots, \vec{v}_{p-1})$, but $\vec{v}_{p-1} \notin \operatorname{span}(\vec{v}_1, \dots, \vec{v}_{p-2})$.

so that

$$\boxed{\vec{v}_p = a_1\vec{v}_1 + \cdots + a_{p-1}\vec{v}_{p-1}} \qquad (1)$$

& apply A:

LHS: $A\vec{v}_p = \lambda_p\vec{v}_p$

RHS: $A(a_1\vec{v}_1 + \cdots + a_{p-1}\vec{v}_{p-1})$

$$= a_1 A\vec{v}_1 + \cdots + a_{p-1}A\vec{v}_{p-1}$$

$$= a_1\lambda_1\vec{v}_1 + \cdots + a_{p-1}\lambda_{p-1}\vec{v}_{p-1}$$

But LHS = RHS:

$$\boxed{\begin{aligned} \lambda_p\vec{v}_p &= a_1\lambda_1\vec{v}_1 + \cdots + a_{p-1}\lambda_{p-1}\vec{v}_{p-1} && (2) \\ &\overset{(1)}{=} \lambda_p(a_1\vec{v}_1 + \cdots + a_{p-1}\vec{v}_{p-1}) && (3) \end{aligned}}$$

$$\Longrightarrow \quad \vec{0} = \lambda_p\vec{v}_p - \lambda_p\vec{v}_p$$

$$\overset{(2)-(3)}{=} (a_1\lambda_1\vec{v}_1 + \cdots + a_{p-1}\lambda_{p-1}\vec{v}_{p-1})$$

plug in (1) for $\vec{v}_p$ in (2) (LHS)

$$- (a_1\lambda_p\vec{v}_1 + \cdots + a_{p-1}\lambda_p\vec{v}_{p-1}) \leftarrow (3)$$

$$= a_1(\lambda_1 - \lambda_p)\vec{v}_1 + \cdots + a_{p-1}(\lambda_{p-1} - \lambda_p)\vec{v}_{p-1}$$

But now we cash in our assumption that <u>the λ_i are all distinct</u>, so $\lambda_1 - \lambda_p \neq 0, \ldots, \lambda_{p-1} - \lambda_p \neq 0$.

But $\vec{v}_{p-1} \notin \operatorname{span}(\vec{v}_1, \ldots, \vec{v}_{p-2}) \Longrightarrow \vec{v}_1, \ldots, \vec{v}_{p-1}$ are linearly <u>independent</u>, so

$$a_1 \underbrace{(\lambda_1 - \lambda_p)}_{\neq 0} = \cdots = a_{p-1} \underbrace{(\lambda_{p-1} - \lambda_p)}_{\neq 0} = 0$$

(*)

$$\Longrightarrow a_1 = \cdots = a_{p-1} = 0$$

$$\overset{(1)}{\Longrightarrow} \vec{v}_p = a_1 \vec{v}_1 + \cdots + a_{p-1} \vec{v}_{p-1} = \vec{0}$$

which is impossible since $\vec{v}_p$ is an eigenvector.

We conclude that the $\vec{v}_i$ are all linearly independent.

<u>QED</u>

the question of equivalence:

The problem with row equivalence is, though it allows us to compute $\det A$ more easily, namely, if

$$E_k \cdots E_1 A = B \in \mathcal{U}_n$$

then

$$\det A = \frac{1}{(\det E_k)\cdots(\det E_1)} \det B$$

where

$$\det E_k = \underset{\substack{\uparrow \\ \text{type I}}}{-1} \text{ or } \underset{\substack{\uparrow \\ \text{type II}}}{c}$$

nevertheless it fails at helping calculate:

$$\det(A - \lambda I) \neq \det(B - \lambda I) \; !$$

or even $\frac{1}{\prod \det E_i} \det(B - \lambda I)$ etc.

ex. 6. Consider the matrix $A = \begin{pmatrix} 1 & -4 & 2 \\ -2 & 8 & -9 \\ -1 & 7 & 0 \end{pmatrix}$ of example 2 (p. 3). Even though

$$E_3 E_2 E_1 A = \begin{pmatrix} 1 & -4 & 2 \\ 0 & 3 & 2 \\ 0 & 0 & 5 \end{pmatrix}$$

yet

$$\det(A - \lambda I) = \det\begin{pmatrix} 1-\lambda & -4 & 2 \\ -2 & 8-\lambda & -9 \\ -1 & 7 & -\lambda \end{pmatrix}$$

$$= (1-\lambda)\det\begin{pmatrix} 8-\lambda & -9 \\ 7 & -\lambda \end{pmatrix}$$
$$+4\det\begin{pmatrix} -2 & -9 \\ -1 & -\lambda \end{pmatrix}$$
$$+2\det\begin{pmatrix} -2 & 8-\lambda \\ -1 & 7 \end{pmatrix}$$

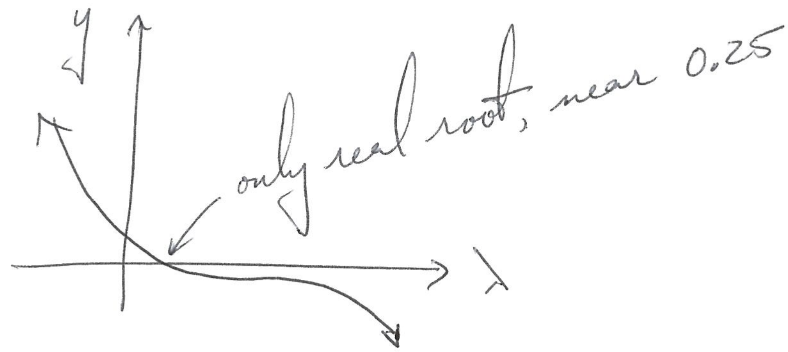
$$= (1-\lambda)\big((8-\lambda)(-\lambda) + 63\big)$$
$$+4(2\lambda - 9)$$
$$+2(\underbrace{-14+8}_{-6}-\lambda)$$

$$= (1-\lambda)(\lambda^2 - 8\lambda + 63)$$
$$+8\lambda - 36$$
$$-12 - 2\lambda$$

$$= -\lambda^3 + 8\lambda^2 - 63\lambda$$
$$+\lambda^2 - \cancel{8\lambda} + 63$$
$$+\cancel{8\lambda} - 36$$

$$\Longrightarrow \quad \det(A-\lambda I) = -\lambda^3 + 9\lambda^2 - 65\lambda + 15$$

But this polynomial doesn't factor completely over $\mathbb{R}$: it has one real root,



Whereas

$$\det(E_3E_2E_1A - \lambda I)$$

$$= \det\begin{pmatrix} 1-\lambda & -4 & 2 \\ 0 & 3-\lambda & 2 \\ 0 & 0 & 5-\lambda \end{pmatrix}$$

$$= (1-\lambda)(3-\lambda)(5-\lambda)$$

has 3 real roots, $\lambda = 1, 3, 5$!

<u>They're totally different polynomials</u> !

The right equivalence will be one which leaves $\det(A-\lambda I)$ unaffected, and fortunately one is ready-to-hand!

Def: Two square matrices $A, B \in M_n(\mathbb{R})$ are said to be similar if there exists an invertible $P \in GL(n,\mathbb{R})$ such that

$$\boxed{A = PBP^{-1}}$$

Prop. Similarity is an equivalence relation on $M_n(\mathbb{R})$, and the equivalence classes are called similarity classes.

pf: (1) Any $A \in M_n(\mathbb{R})$ is similar to itself,

(reflexivity) $\left\{ A \sim A \text{ since } A = I_n A I_n^{-1} \quad \& \ I_n \in GL(n, \mathbb{R}) \right.$

(2) If $A \sim B$, then $A = PBP^{-1}$ for

(symmetry) some $P \in GL(n, \mathbb{R})$. But then $P^{-1} \in GL(n, \mathbb{R})$, too, &

$$B = (P^{-1})AP$$
$$= (P^{-1})A(P^{-1})^{-1},$$

so $B \sim A$.

(3) If $A \sim B$ and $B \sim C$, then

(transitivity)

$$\left.\begin{aligned} A &= PBP^{-1} \\ B &= QCQ^{-1} \end{aligned}\right\} \text{ for some } P, Q \in GL(n, \mathbb{R})$$

$$\Longrightarrow A = PBP^{-1}$$
$$= P(QCQ^{-1})P^{-1} \qquad \& \ PQ \in GL(n, \mathbb{R})$$
$$\Rightarrow A \sim C \qquad = (PQ)C(PQ)^{-1} \qquad \text{QED}$$

<u>Remark</u>: Similarity, $A \sim B \overset{\text{def}}{\Longleftrightarrow} A = PBP^{-1}$, has a ~~...~~ special meaning for us, since

$$\boxed{\begin{array}{l} P = \begin{pmatrix} | & & | \\ \vec{P}_1 & \cdots & \vec{P}_n \\ | & & | \end{pmatrix} \in GL(n, \mathbb{R}) \\ \iff \rho = (\vec{P}_1, \dots, \vec{P}_n) \text{ is a basis for } \mathbb{R}^n \end{array}}$$

What kind of basis is ρ?

<u>Prop. 1</u>

$$\boxed{A \sim B \implies \det(A - \lambda I) = \det(B - \lambda I)}$$

pf: $A \sim B \Rightarrow A = PBP^{-1} \implies$

$$\det(A - \lambda I) = \det(PBP^{-1} - \lambda(PP^{-1}))$$
$$= \det(P(B - \lambda I)P^{-1})$$
$$= \cancel{\det P} \det(B - \lambda I) \cancel{\det P^{-1}}$$

QED

<u>Cor. 1</u> Let β, γ be two bases for $\mathbb{R}^n$ (or $\mathbb{C}^n$), & note that for any $A \in M_n(\mathbb{R})$ (or $\mathbb{C}$),

$$\boxed{\det [A]_\beta = \det [A]_\gamma = \det A}$$

<u>pf:</u> Apply Prop. 1 to

$$[A]_\gamma = M_{\beta\gamma} [A]_\beta M_{\beta\gamma}^{-1}$$

which says

$$[A]_\gamma \sim [A]_\beta$$

Apply it also to $A = [A]_\sigma = M_{\beta\sigma} [A]_\beta M_{\beta\sigma}^{-1}$.

<u>QED</u>

<u>Def.</u> For <u>any</u> linear operator $T \in \mathcal{L}(V)$ on a <u>finite-dimensional vector space</u> V,

$$\boxed{\det T \overset{\text{def}}{=} \det [T]_\beta \text{ for any basis } \beta \text{ for } V}$$

Returning to $A \sim B \overset{\text{def}}{\iff} A = PBP^{-1}$

for some $P = \begin{pmatrix} | & & | \\ \vec{p}_1 & \cdots & \vec{p}_n \\ | & & | \end{pmatrix} \in GL(n, \mathbb{R}) \iff$

$\rho = (\vec{p}_1, \dots, \vec{p}_n)$ a basis for $\mathbb{R}^n \iff P = M_{\rho\sigma}$

$\iff A = PBP^{-1} = M_{\rho\sigma} B M_{\rho\sigma}^{-1}$

$\iff B = [A]_\rho$ & $A = [B]_\sigma$, since

$$
\left.\begin{aligned}
[B]_\sigma &= M_{\rho\sigma} [B]_\rho M_{\rho\sigma}^{-1} \\
&= PBP^{-1} \\
&= A
\end{aligned}\right\} \text{? yes}
$$

&
$$
\begin{aligned}
[A]_\rho &= M_{\sigma\rho} [A]_\sigma M_{\sigma\rho}^{-1} \\
&= M_{\rho\sigma}^{-1} A M_{\rho\sigma} \\
&= P^{-1} A P \\
&= B
\end{aligned}
$$

<u>Suppose, now, that all $\vec{p}_i$ are eigenvectors of A</u>:

$$\boxed{A\vec{p}_i = \lambda_i \vec{p}_i}$$

Not all λ_i are assumed to be distinct (e.g. $A = I_n$ has $\lambda_i = 1$ for all i.), but since $P^{-1}\vec{p}_i = \vec{e}_i$ (bec. $P^{-1}P = I_n$)

$$\boxed{\begin{aligned} P^{-1}AP &= P^{-1}\begin{pmatrix} | & & | \\ A\vec{p}_1 & \cdots & A\vec{p}_n \\ | & & | \end{pmatrix} \\ &= P^{-1}\left(\lambda_1\vec{p}_1 \cdots \lambda_n\vec{p}_n\right) \\ &= \left(\lambda_1 P^{-1}\vec{p}_1 \cdots \lambda_n P^{-1}\vec{p}_n\right) \\ &= \left(\lambda_1\vec{e}_1 \cdots \lambda_n\vec{e}_n\right) \\ &= \begin{pmatrix} \lambda_1 & & 0 \\ & \ddots & \\ 0 & & \lambda_n \end{pmatrix} \end{aligned}}$$

Let $D = \begin{pmatrix} \lambda_1 & & 0 \\ & \ddots & \\ 0 & & \lambda_n \end{pmatrix}$, so that we may write

$$\boxed{P^{-1}AP = D}$$

where $\boxed{P = (\vec{p}_1 \cdots \vec{p}_n) \in GL(n, \mathbb{R})}$

Thus,

$\boxed{\rho = (\vec{P}_1, \dots, \vec{P}_n)}$ basis of eigenvectors of A for $\mathbb{R}^n$, w/ corresponding eigenvalues $\lambda_1, \dots, \lambda_n$ (not nec. distinct)

$\Longrightarrow$ $\boxed{P = M_{\rho\sigma}}$ and $\boxed{P^{-1}AP = D}$

where $\boxed{D = \begin{pmatrix} \lambda_1 & & 0 \\ & \ddots & \\ 0 & & \lambda_n \end{pmatrix} = [A]_\rho}$

$\Longleftrightarrow$ $A = PDP^{-1}$ i.e. $A \sim D$

similar to

Def: $A \in M_n(\mathbb{R})$ is said to be *diagonalizable* if $A \sim D$, i.e. A is similar to a diagonal matrix D, via $P = M_{po} \in GL(n, \mathbb{R})$.

If A is diagonalizable,

$$A = PDP^{-1}$$

$$\iff D = P^{-1}AP$$

$$\iff AP = PD$$

$$AP = (A\vec{p}_1 \cdots A\vec{p}_n)$$

$$PD = P(\lambda_1\vec{e}_1 \cdots \lambda_n\vec{e}_n) = (\lambda_1 P\vec{e}_1 \cdots \lambda_n P\vec{e}_n) = (\lambda_1\vec{p}_1 \cdots \lambda_n\vec{p}_n)$$

Then

$$\underline{A\vec{p}_i = \lambda_i\vec{p}_i}$$

We have just proven:

<u>Thm. 1</u> $A \in M_n(\mathbb{R})$ is diagonalizable $\left(A \sim D \overset{\text{def}}{\iff} A = PDP^{-1} \text{ for some } P \in GL(n, \mathbb{R}) \right)$ <u>iff</u> $P = (\vec{P}_1, \dots, \vec{P}_n)$ is a basis for $\mathbb{R}^n$ (or $\mathbb{C}^n$) consisting of eigenvectors of A corresponding to the eigenvalues λ_i appearing along the diagonal of $D = \begin{pmatrix} \lambda_1 & & 0 \\ & \ddots & \\ 0 & & \lambda_n \end{pmatrix}$. $\boxtimes$