Examples and Exercises in Coordinate Representations

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1 Examples and Exercises in $\mathbb{R}^2$

Example 1.1 Find the β-representation of the $\pi/2$-rotation operator, which in standard coordinates appears as

$$[R_{\pi/2}]_\sigma = \begin{pmatrix} \cos(\pi/2) & -\sin(\pi/2) \\ \sin(\pi/2) & \cos(\pi/2) \end{pmatrix} = \begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix} \in \mathrm{GL}(2,\mathbb{R})$$

where

$$\beta = (\mathbf{b}_1, \mathbf{b}_2) = \left(\begin{pmatrix} 2 \\ 1 \end{pmatrix}, \begin{pmatrix} -1 \\ 1 \end{pmatrix} \right)$$

Solution: Use the change-of-coordinates formula for operators

$$[T]_\gamma = M_{\beta,\gamma}[T]_\beta M_{\beta,\gamma}^{-1}$$

given in Proposition 6.18, 'Bases, Coordinates and Representations' with $T = R_{\pi/2}$, replacing β with σ and γ with β:

$$\begin{aligned} [R_{\pi/2}]_\beta &= M_{\sigma,\beta}[R_{\pi/2}]_\sigma M_{\sigma,\beta}^{-1} \\ &= M_{\beta,\sigma}^{-1}[R_{\pi/2}]_\sigma M_{\beta,\sigma} \\ &= \begin{pmatrix} 2 & -1 \\ 1 & 1 \end{pmatrix}^{-1} \begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix} \begin{pmatrix} 2 & -1 \\ 1 & 1 \end{pmatrix} \\ &= \frac{1}{3}\begin{pmatrix} 1 & 1 \\ -1 & 2 \end{pmatrix} \begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix} \begin{pmatrix} 2 & -1 \\ 1 & 1 \end{pmatrix} \\ &= \boxed{\frac{1}{3}\begin{pmatrix} 1 & -2 \\ 5 & -1 \end{pmatrix}} \end{aligned}$$

Notice how different the rotation matrix looks in non-standard coordinates! ■

Exercise 1.2 Find the γ-representation of the $\pi/2$ rotation operator $R_{\pi/2} \in \mathrm{GL}(\mathbb{R}^2)$, where

$$\gamma = (\mathbf{c}_1, \mathbf{c}_2) = \left(\begin{pmatrix} 3 \\ 2 \end{pmatrix}, \begin{pmatrix} 1 \\ 0 \end{pmatrix} \right)$$ ■

Example 1.3 Compute the β-representation of the reflection across the line $y = -2x$, with

$$\beta = (\mathbf{b}_1, \mathbf{b}_2) = \left(\begin{pmatrix} 2 \\ 1 \end{pmatrix}, \begin{pmatrix} -1 \\ 1 \end{pmatrix} \right)$$

Solution: Let us first find a *natural* basis δ for R_ℓ. The line

$$\ell = \left\{ \mathbf{x} = \begin{pmatrix} x \\ y \end{pmatrix} \in \mathbb{R}^2 \;\middle|\; y = -2x \right\}$$

has basis vector $\mathbf{d}_1 = \begin{pmatrix} 1 \\ -2 \end{pmatrix}$, which can be gotten from the equation $y = -2x$ by plugging in $x = 1$. Rotate it by $\pi/2$ to get a basis vector for the orthogonal line $\ell^\perp$, $y = \frac{1}{2}x$:

$$\mathbf{d}_2 = R_{\pi/2}\mathbf{d}_1 = \begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix} \begin{pmatrix} 1 \\ -2 \end{pmatrix} = \begin{pmatrix} 2 \\ 1 \end{pmatrix}$$

Then we have a basis for $\mathbb{R}^2$ suited to R_ℓ:

$$\boxed{\delta = (\mathbf{d}_1, \mathbf{d}_2) = \left(\begin{pmatrix} 1 \\ -2 \end{pmatrix}, \begin{pmatrix} 2 \\ 1 \end{pmatrix} \right)}$$

Since $R_\ell(\mathbf{d}_1) = \mathbf{d}_1$ and $R_\ell(\mathbf{d}_2) = -\mathbf{d}_2$, the δ-representation of R_ℓ is simple:

$$\boxed{[R_\ell]_\delta = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}}$$

Now use the change-of-coordinates formula for operators given in Proposition 6.18, 'Bases, Coordinates and Representations',

$$[T]_\gamma = M_{\beta,\gamma}[T]_\beta M_{\beta,\gamma}^{-1}$$

replacing γ with β and β with δ, along with its consequence, Proposition 6.19, which allows us to compute $M_{\delta,\beta}$ in terms of the standard basis σ:

$$M_{\delta,\beta} = M_{\sigma,\beta} M_{\delta,\sigma} = M_{\beta,\sigma}^{-1} M_{\delta,\sigma}$$

Here we go:

$$\begin{aligned}
[R_\ell]_\beta &= M_{\delta,\beta}[R_\ell]_\delta M_{\delta,\beta}^{-1} && (\textit{Proposition 6.18}) \\
&= M_{\beta,\sigma}^{-1} M_{\delta,\sigma}[R_\ell]_\delta \left(M_{\beta,\sigma}^{-1} M_{\delta,\sigma}\right)^{-1} && (\textit{Proposition 6.19}) \\
&= M_{\beta,\sigma}^{-1} M_{\delta,\sigma}[R_\ell]_\delta M_{\delta,\sigma}^{-1} M_{\beta,\sigma} \\
&= \begin{pmatrix} 2 & -1 \\ 1 & 1 \end{pmatrix}^{-1} \begin{pmatrix} 1 & 2 \\ -2 & 1 \end{pmatrix} \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix} \begin{pmatrix} 1 & 2 \\ -2 & 1 \end{pmatrix}^{-1} \begin{pmatrix} 2 & -1 \\ 1 & 1 \end{pmatrix} \\
&= \boxed{\begin{pmatrix} -1 & \frac{2}{5} \\ 0 & 1 \end{pmatrix}}
\end{aligned}$$

Verification:

How do we know if we got it right? There is no solution manual, I just made up the example. Well, we know that $R_\ell(\mathbf{d}_1) = \mathbf{d}_1$ and $R_\ell(\mathbf{d}_2) = -\mathbf{d}_2$ (that's how we got the δ-representation of R_ℓ above, after all): Well, we have Theorem 6.16 in 'Bases, Coordinates and Representations,' which tells us $[T(\mathbf{v})]_\beta = [T]_\beta[\mathbf{v}]_\beta$. Let's use it on R_ℓ and the basis vectors $\mathbf{d}_i$: if we computed $[R_\ell]_\beta$ correctly, the RHS should equal the LHS. The LHS for each $\mathbf{d}_i$ is

$$\begin{aligned}[R_\ell(\mathbf{d}_1)]_\beta = [\mathbf{d}_1]_\beta = M_{\sigma,\beta}[\mathbf{d}_1]_\sigma = M_{\beta,\sigma}^{-1}\mathbf{d}_1 \\ = \begin{pmatrix} 2 & -1 \\ 1 & 1 \end{pmatrix}^{-1} \begin{pmatrix} 1 \\ -2 \end{pmatrix} = \frac{1}{3}\begin{pmatrix} 1 & 1 \\ -1 & 2 \end{pmatrix}\begin{pmatrix} 1 \\ -2 \end{pmatrix} = -\frac{1}{3}\begin{pmatrix} 1 \\ 5 \end{pmatrix}\end{aligned}$$

and similarly

$$\begin{aligned}[R_\ell(\mathbf{d}_2)]_\beta = [-\mathbf{d}_2]_\beta = -M_{\sigma,\beta}[\mathbf{d}_2]_\sigma = -M_{\beta,\sigma}^{-1}\mathbf{d}_2 \\ = -\begin{pmatrix} 2 & -1 \\ 1 & 1 \end{pmatrix}^{-1} \begin{pmatrix} 2 \\ 1 \end{pmatrix} = -\frac{1}{3}\begin{pmatrix} 1 & 1 \\ -1 & 2 \end{pmatrix}\begin{pmatrix} 2 \\ 1 \end{pmatrix} = \begin{pmatrix} -1 \\ 0 \end{pmatrix}\end{aligned}$$

This, incidentally, gives us $\boxed{[\mathbf{d}_1]_\beta = -\frac{1}{3}\begin{pmatrix} 1 \\ 5 \end{pmatrix}}$ and $\boxed{[\mathbf{d}_2]_\beta = \begin{pmatrix} 1 \\ 0 \end{pmatrix}}$. Let's make sure these are correct first:

$$-\frac{1}{3}\mathbf{b}_1 - \frac{5}{3}\mathbf{b}_2 = -\frac{1}{3}\begin{pmatrix} 2 \\ 1 \end{pmatrix} - \frac{5}{3}\begin{pmatrix} -1 \\ 1 \end{pmatrix} = \begin{pmatrix} 1 \\ -2 \end{pmatrix} = \mathbf{d}_1 \quad \checkmark$$

and secondly,

$$1\mathbf{b}_1 + 0\mathbf{b}_2 = \begin{pmatrix} 2 \\ 1 \end{pmatrix} = \mathbf{d}_2 \quad \checkmark$$

OK! So far so good. Now, if we got $[R_\ell]_\beta$ right, we should multiply out the RHS for each $\mathbf{d}_i$ and get back what we computed for the LHS:

$$[R_\ell]_\beta[\mathbf{d}_1]_\beta = \begin{pmatrix} -1 & \frac{2}{5} \\ 0 & 1 \end{pmatrix}\left(-\frac{1}{3}\begin{pmatrix} 1 \\ 5 \end{pmatrix}\right) = -\frac{1}{3}\begin{pmatrix} 1 \\ 5 \end{pmatrix} = [\mathbf{d}_1]_\beta = [R_\ell(\mathbf{d}_1)]_\beta \quad \checkmark$$

and

$$[R_\ell]_\beta[\mathbf{d}_2]_\beta = \begin{pmatrix} -1 & \frac{2}{5} \\ 0 & 1 \end{pmatrix}\begin{pmatrix} 1 \\ 0 \end{pmatrix} = \begin{pmatrix} -1 \\ 0 \end{pmatrix} = [R_\ell(\mathbf{d}_2)]_\beta \quad \checkmark$$

Exercise 1.4 Compute the γ-representation of the reflection across the line $y = -2x$, where

$$\gamma = (\mathbf{c}_1, \mathbf{c}_2) = \left(\begin{pmatrix} 3 \\ 2 \end{pmatrix}, \begin{pmatrix} 1 \\ 0 \end{pmatrix}\right) \qquad \blacksquare$$