Quiz 8

1. Let $\mathbf{F}(x,y) = \langle 3x^2, 4y^3 \rangle$.

   (a) Compute directly $\int_{C_1} \mathbf{F} \cdot d\mathbf{r}$ where C_1 is the curve around the top of the unit circle from $(1,0)$ to $(-1,0)$.

   Parametrize C_1: $\mathbf{r}_1(t) = \langle \cos t, \sin t \rangle$, $0 \le t \le \pi$. Then,

$$\begin{aligned}
\int_{C_1} \mathbf{F} \cdot d\mathbf{r} &= \int_0^{\pi} \mathbf{F}\big(\mathbf{r}_1(t)\big) \cdot d\mathbf{r}_1'(t)\, dt \\
&= \int_0^{\pi} \langle 3\cos^2 t, 4\sin^3 t \rangle \cdot \langle -\sin t, \cos t \rangle\, dt \\
&= \int_0^{\pi} -3\cos^2 t \sin t + 4\sin^3 t \cos t\, dt \\
&= -3\int_0^{\pi} \cos^2 t \sin t\, dt + 4\int_0^{\pi} \sin^3 t \cos t\, dt \\
&= 3\int_1^{-1} u^2\, du + 4\int_0^0 u^3\, du \\
&= \boxed{-2}
\end{aligned}$$

   (b) Compute directly $\int_{C_2} \mathbf{F} \cdot d\mathbf{r}$ where C_2 is the straight line curve from $(1,0)$ to $(-1,0)$.

   Parametrize C_2: $\mathbf{r}_2(t) = \langle 1,0 \rangle + t(\langle -1,0 \rangle - \langle 1,0 \rangle) = \langle 1-2t, 0 \rangle$, $0 \le t \le 1$. Then,

$$\begin{aligned}
\int_{C_2} \mathbf{F} \cdot d\mathbf{r} &= \int_0^{1} \mathbf{F}\big(\mathbf{r}_2(t)\big) \cdot d\mathbf{r}_2'(t)\, dt \\
&= \int_0^{1} \langle 3(1-2t)^2, 4 \cdot 0^3 \rangle \cdot \langle -2, 0 \rangle\, dt \\
&= -6\int_0^{1} 1 - 4t + 4t^2\, dt \\
&= -6\Big[t - 2t + \frac{4}{3}t^3\Big]_0^1 \\
&= -6\Big[1 - 2 + \frac{4}{3}\Big] \\
&= \boxed{-2}
\end{aligned}$$

   (c) Show that $\mathbf{F}$ is a continuous gradient vector field, that is $\mathbf{F} = \nabla f$ for some scalar function $f : \mathbb{R}^2 \to \mathbb{R}$, and use this along with the Fundamental Theorem of Line Integrals to compute $\int_C \mathbf{F} \cdot d\mathbf{r}$ (Thus, $\mathbf{F}$ is a conservative, or path-independent, vector field.)

   If there is such a scalar function $f : \mathbb{R}^2 \to \mathbb{R}$, it will have to satisfy $f_x = 3x^2$ and $f_y = 4y^3$. Well, clearly $f(x,y) = x^3 + y^4$ is one such function, and so $\mathbf{F} = \nabla f$. That $\mathbf{F}$ is continuous follows from the fact that it's component functions $F_1(x,y) = 3x^2$ and $F_2(x,y) = 4y^3$ are polynomials, which are continuous everywhere in $\mathbb{R}^2$. Consequently, by the Fundamental Theorem of Calculus for Line Integrals we have that

$$\int_C \mathbf{F} \cdot d\mathbf{r} = \int_C \nabla f \cdot d\mathbf{r} = f(-1,0) - f(1,0) = \big((-1)^3 + 0^4\big) - (1^3 + 0^2) = \boxed{-2}$$

2. Let $\mathbf{F} = \langle y, -x \rangle$.

   (a) Compute the integral $\int_C \mathbf{F} \cdot d\mathbf{r}$ directly, where C is the counterclockwise curve along the unit circle starting at $(1, 0)$.

   As usual, we parametrize C first: $\mathbf{r}(t) = \langle \cos t, \sin t \rangle$, $0 \leq t \leq 2\pi$. Then

$$\int_C \mathbf{F} \cdot d\mathbf{r} = \int_0^{2\pi} \mathbf{F}\big(\mathbf{r}(t)\big) \cdot \mathbf{r}'(t)\, dt = \int_0^{2\pi} \langle \sin t, -\cos t \rangle \cdot \langle -\sin t, \cos t \rangle\, dt$$

$$= \int_0^{2\pi} -\sin^2 t - \cos^t\, dt = -\int_0^{2\pi} 1\, dt = \boxed{-2\pi}$$

   (b) Use Green's Theorem to compute the integral $\int_C \mathbf{F} \cdot d\mathbf{r}$, where C is the counterclockwise curve along the unit circle starting at $(1, 0)$, bounding the unit disk $\mathbb{D}^2$.

   Since $F_1(x, y) = y$ and $F_2(x, y) = -x$, we have

$$\int_C \mathbf{F} \cdot d\mathbf{r} = \iint_{\mathbb{D}^2} \frac{\partial F_2}{\partial x} - \frac{\partial F_1}{\partial y}\, dA = \iint_{\mathbb{D}^2} -1 - 1 dA = -2 \iint_{\mathbb{D}^2} 1\, dA = \boxed{-2\pi}$$