Quiz 7

1. Let C be the straight line path from $(0,0)$ to $(5,5)$ and let $\mathbf{F} = (y - x + 2)\mathbf{i} + (\sin(y - x) - 2)\mathbf{j}$.

   (a) At each point of C, what angle does $\mathbf{F}$ make with the tangent vector to C?

   First, let's parametrize C: $\mathbf{r}(t) = \langle 0,0\rangle + t(\langle 5,5\rangle - \langle 0,0\rangle) = \langle 5t, 5t\rangle$, where $0 \le t \le 1$. Consequently, $\mathbf{r}'(t) = \langle 5,5\rangle$, and so

   $$\mathbf{F}\big(\mathbf{r}(t)\big) \cdot \mathbf{r}'(t) = \langle 5t - 5t + 2, \sin(5t - 5t) - 2\rangle \cdot \langle 5,5\rangle = \langle 2,-2\rangle \cdot \langle 5,5\rangle = 10 - 10 = 0$$

   and so the angle θ between $\mathbf{F}(\mathbf{r}(t))$ and $\mathbf{r}'(t)$ is $\boxed{\theta = \frac{\pi}{2}}$ (they're orthogonal).

   (b) Evaluate $\int_C \mathbf{F} \cdot d\mathbf{r}$.

   $$\int_C \mathbf{F} \cdot d\mathbf{r} = \int_0^1 \mathbf{F}\big(\mathbf{r}(t)\big) \cdot \mathbf{r}'(t)\, dt = \int_0^1 0\, dt = \boxed{0}$$

2. Let $\mathbf{F} = -y\mathbf{i} + x\mathbf{j}$ and let C be the unit circle oriented counterclockwise.

   (a) Show that $\mathbf{F}$ has constant magnitude of 1 on C.

   First, let's parametrize C. Since it's a circle of radius 1, $\mathbf{r}(t) = \langle \cos t, \sin t\rangle$ where $0 \le t \le 2\pi$, and so

   $$\|\mathbf{F}\big(\mathbf{r}(t)\big)\| = \sqrt{\langle -\sin t, \cos t\rangle} = \sqrt{\sin^2 t + \cos^2 t} = 1$$

   (b) Show that $\mathbf{F}$ is always tangent to the circle.

   First, a tangent vector to the circle is $\mathbf{r}'(t) = \langle -\sin t, \cos t\rangle$. But this is precisely $\mathbf{F}(\mathbf{r}(t))$!

   (c) Show that $\int_C \mathbf{F} \cdot d\mathbf{r}$ equals the length of C.

   $$\begin{aligned}
   \int_C \mathbf{F} \cdot d\mathbf{r} &= \int_0^{2\pi} \mathbf{F}\big(\mathbf{r}(t)\big) \cdot \mathbf{r}'(t)\, dt \\
   &= \int_0^{2\pi} \langle -\sin t, \cos t\rangle \cdot \langle -\sin t, \cos t\rangle\, dt \\
   &= \int_0^{2\pi} \|\langle -\sin t, \cos t\rangle\|^2\, dt \\
   &= \int_0^{2\pi} 1\, dt \\
   &= 2\pi
   \end{aligned}$$