Quiz 4

1. Find and classify the critical points of $f(x, y) = \ln(x^2 + y^2 + 1)$.

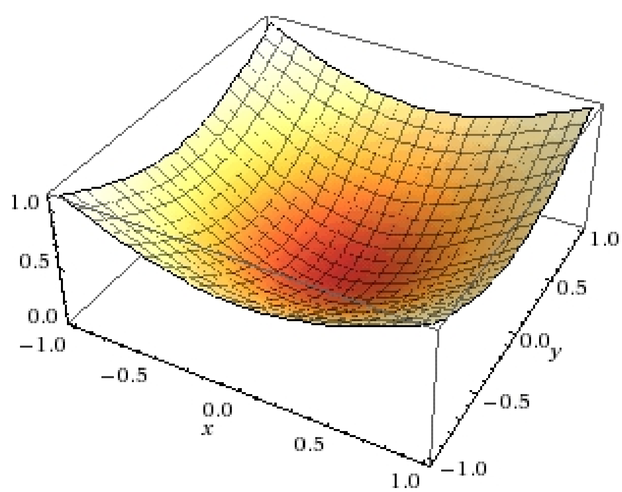
   Let's compute some partial derivatives:

$$
\begin{aligned}
f_x(x,y) &= \frac{2x}{x^2+y^2+1} \\
f_{xx}(x,y) &= \frac{-2x^2+2y^2+2}{(x^2+y^2+1)^2} \\
f_y(x,y) &= \frac{2y}{x^2+y^2+1} \\
f_{yy}(x,y) &= \frac{2x^2-2y^2+2}{(x^2+y^2+1)^2} \\
f_{xy}(x,y) &= \frac{-2xy}{(x^2+y^2+1)^2}
\end{aligned}
$$

   OK, so $f_x(x, y) = 0$ implies $x = 0$ and $f_y(x, y) = 0$ implies $y = 0$, so in order that both be zero, i.e. that $\nabla f(x, y) = (0, 0)$, we must have $(x, y) = (0, 0)$, and this is the only critical point. From the above calculations we see that $f_{xx}(0, 0) = 2$, $f_{yy}(0, 0) = 2$ and $f_{xy}(0, 0) = f_{yx}(0, 0) = 0$, so

$$
D = \det Hf(0,0) = \det \begin{bmatrix} f_{xx}(0,0) & f_{xy}(0,0) \\ f_{yx}(0,0) & f_{yy}(0,0) \end{bmatrix} = \det \begin{bmatrix} 2 & 0 \\ 0 & 2 \end{bmatrix} = 4 > 0
$$

   Since $f_{xx}(0, 0) = 2 > 0$, we see that $(0, 0)$ is a local minimum. Indeed, here is a graph of this function around $(0, 0)$:



2. Find the absolute maximum and absolute minimum of $f(x, y) = \dfrac{1}{\|(x, y)\|}$ on $1 \le \|(x, y)\| \le 2$.

It's easier to do this problem if you switch to polar coordinates, since when we let $x = r\cos\theta$ and $y = r\sin\theta$, then

$$\|(x, y)\| = \sqrt{x^2 + y^2} = \sqrt{r^2(\sin^2\theta + \cos^2\theta)} = r$$

so that the function is in fact

$$f(r, \theta) = \frac{1}{r}$$

where $1 \le r \le 2$. Obviously this function will have no critical points and it will achieve its maximum on the circle $r = 1$.

But let us check this directly: $f_x = \frac{-x}{(x^2+y^2)^{3/2}}$ and $f_y = \frac{-y}{(x^2+y^2)^{3/2}}$, so that $\nabla f(0, 0) = (0, 0)$ only when $(x, y) = (0, 0)$, but this lies outside of the annulus $1 \le \|(x, y)\| \le 2$. Hence f has no critical points in this region. We have only to check the boundary, (x, y) on the circles of radius 1 and 2, and clearly there $f(x, y) = 1$ and $1/2$ respectively. Thus the maximum of f occurs on the circle of radius 1, $\|(x, y)\| = 1$, where f has a value of 1.