Quiz 3

1. Suppose a differentiable real-valued function $f(x,y)$ satisfies the equation

$$af_x + bf_y = c$$

for all x and y, where a, b and c are arbitrary constants. Introduce the new variables

$$w = bx - ay \qquad \text{and} \qquad z = y$$

(so that $x = \frac{1}{b}(w + az)$ and $y = z$), and define $g(w,z) \equiv f(x,y) = f\big(\frac{1}{b}(w+az), z\big)$ (*Note*: this implies $f(x,y) = g(w,z) = g(bx - ay, y)$). Use the chain rule to show that

$$bg_z = c$$

[*Hint*: Start with the first equation and derive the second from it.]

We just use the chain rule, noting that $f = g$:

$$\begin{aligned} c &= af_x + bf_y \\ &= ag_x + bg_y \\ &= a(g_w w_x + g_z z_x) + b(g_w w_y + g_z z_y) \\ &= a(bg_w + 0g_z) + b(-ag_w + 1g_z) \\ &= bg_z \end{aligned}$$

Remark: *If we wanted to be more explicit about our functions and use matrices, we would let $\varphi : \mathbb{R}^2 \to \mathbb{R}^2$ be the change-of-variable function*

$$(w,z) = \varphi(x,y) = (bx - ay, y)$$

sending (x,y) coordinates to (w,z) coordinates, so that $w = \varphi_1(x,y) = bx - ay$ and $z = \varphi_2(x,y) = y$. Then,

$$D\varphi(x,y) = \begin{bmatrix} \varphi_{1x}(x,y) & \varphi_{1y}(x,y) \\ \varphi_{2x}(x,y) & \varphi_{2y}(x,y) \end{bmatrix} = \begin{bmatrix} b & -a \\ 0 & 1 \end{bmatrix}$$

Now, the key observation is that $f(x,y) = (g \circ \varphi)(x,y)$, and this allows us to use the chain rule:

$$\begin{aligned} \begin{bmatrix} f_x(x,y) & f_y(x,y) \end{bmatrix} &= Df(x,y) \\ &= D(g\circ\varphi)(x,y) \\ &= Dg(\varphi(x,y)) \cdot D\varphi(x,y) \\ &= \begin{bmatrix} g_w(\varphi(x,y)) & g_z(\varphi(x,y)) \end{bmatrix} \begin{bmatrix} \varphi_{1x}(x,y) & \varphi_{1y}(x,y) \\ \varphi_{2x}(x,y) & \varphi_{2y}(x,y) \end{bmatrix} \\ &= \begin{bmatrix} g_w(w,z) & g_z(w,z) \end{bmatrix} \begin{bmatrix} b & -a \\ 0 & 1 \end{bmatrix} \\ &= \begin{bmatrix} bg_w(w,z) & -ag_w(w,z) + g_z(w,z) \end{bmatrix} \end{aligned}$$

Comparing components, we get

$$\begin{aligned} f_x(x,y) &= bg_w(w,z) \\ f_y(x,y) &= -ag_w(w,z) + g_z(w,z) \end{aligned}$$

and therefore, plugging these into $c = af_x + bf_y$ we get our result:

$$c = af_x(x,y) + bf_y(x,y) = a\big(bg_w(w,z)\big) + b\big(-ag_w(w,z) + g_z(w,z)\big) = bg_z(w,z)$$

2. (*Implicit Differentiation in From a Multivariate Viewpoint*) Suppose that $y(x)$ is defined implicitly by $G(x, y(x)) = 0$, where G is a given function of two variables.

   (a) If $y(x)$ and G are differentiable and $\partial G/\partial y \neq 0$, show that

$$\frac{dy}{dx} = -\frac{\partial G/\partial x}{\partial G/\partial y}$$

Taking the partial derivative of both sides of $0 = G(x, y(x))$ with respect to x and using the chain rule we get

$$0 = \frac{\partial}{\partial x}0 = \frac{\partial}{\partial x}G(x, y(x)) = \frac{\partial G}{\partial x}\frac{dx}{dx} + \frac{\partial G}{\partial y}\frac{dy}{dx} = \frac{\partial G}{\partial x} + \frac{\partial G}{\partial y}\frac{dy}{dx}$$

so solve this for $\frac{dy}{dx}$ and you're done:

$$\boxed{\frac{dy}{dx} = -\frac{\partial G/\partial x}{\partial G/\partial y}}$$

   (b) Use part (a) to find $\dfrac{dy}{dx}$ if $x^2 + y^3 + e^y = 0$. For what points (x, y) is the assumption $\partial G/\partial y \neq 0$ valid here?

Here $G(x, y(x)) = x^2 + y^3 + e^y$, so $\partial G/\partial x = 2x$ and $\partial G/\partial y = 3y^2 + e^y$. Consequently, by part (a)

$$\boxed{\frac{dy}{dx} = -\frac{\partial G/\partial x}{\partial G/\partial y} = -\frac{2x}{3y^2 + e^y}}$$

Now, this will only work if we're not dividing by 0, of course. But $3y^2 + e^y$ is strictly greater than 0 for all $y \in \mathbb{R}$, so we needen't worry. Thus, this works for all $(x, y) \in \mathbb{R}^2$.