Quiz 2

1. For real numbers a, b and t such that $2a > b$ and $t > 0$, let $\mathbf{u} = (ta, t, t)$, and $\mathbf{v} = (tb, 2t, 2t)$. Find t such that $\|\mathbf{u} \times \mathbf{v}\| = \sqrt{2}$.

   Since

   $$2 = \|\mathbf{u} \times \mathbf{v}\|^2 = \|(0, bt^2 - 2at^2, 2at^2 - bt^2)\|^2 = t^4(b - 2a)^2 + t^4(2a - b)^2 = 2t^4(2a - b)^2$$

   we have

   $$t^4 = \frac{1}{(2a - b)^2}$$

   and so, taking the positive 4th root (since $t > 0$), we get

   $$\boxed{t = \frac{1}{\sqrt{2a - b}}}$$

2. Find an equation of the plane that contains the line $y = -3x + 4$ in the plane $z = 2$, and goes throught the point $(1, 3, 7)$.

   Find three points P, Q and R in the plane first: we are given $z = 2 \implies y = -3x + 4$, so setting $x = 0$ and 1 gives $P = (0, 4, 2)$ and $Q = (1, 1, 2)$. $R = (1, 3, 7)$ is already given to us. Then let

   $$\mathbf{u} = \vec{PQ} = \langle 1, -3, 0 \rangle \qquad \text{and} \qquad \mathbf{v} = \vec{PR} = \langle 1, -1, 5 \rangle$$

   Taking their cross product gives us a normal to the plane:

   $$\mathbf{n} = \mathbf{u} \times \mathbf{v} = \langle -15, -5, 2 \rangle$$

   and from this we get the equation of the plane, picking, say P, as our representative point:

   $$\boxed{-15(x - 0) - 5(y - 4) + 2(z - 2) = 0} \quad \text{or} \quad \boxed{15x + 5y - 2z = 16}$$

3. The points $(5, 0, 0)$, $(0, -3, 0)$ and $(0, 0, 2)$ form a triangle.

   (a) Find the lengths of the sides of the triangle.

   $$a = \sqrt{(5 - 0)^2 + (0 - (-3))^2 + (0 - 0)^2} = \sqrt{34}$$
   $$b = \sqrt{(5 - 0)^2 + (0 - 0)^2 + (0 - 2)^2} = \sqrt{29}$$
   $$c = \sqrt{(0 - 0)^2 + (-3 - 0)^2 + (0 - 2)^2} = \sqrt{13}$$

   (b) Find the area of the triangle.

   Let $\mathbf{u}$ be the vector from $(5, 0, 0)$ to $(0, -3, 0)$ and $\mathbf{v}$ be the vector from $(5, 0, 0)$ to $(0, 0, 2)$, so $\mathbf{u} = \langle 5, 3, 0 \rangle$, $\mathbf{v} = \langle 5, 0, -2 \rangle$. Then, $\mathbf{u} \times \mathbf{v} = \langle -6, 10, -15 \rangle$, and

   $$\boxed{A = \frac{1}{2}\|\mathbf{u} \times \mathbf{v}\| = \frac{19}{2}}$$

   (c) Find the equation of the plane containing the triangle.

   Using $\mathbf{u} = \langle 5, 3, 0 \rangle$, $\mathbf{v} = \langle 5, 0, -2 \rangle$ and $\mathbf{u} \times \mathbf{v} = \langle -6, 10, -15 \rangle$ from part (b), and knowing that $\mathbf{u} \times \mathbf{v}$ is normal to the plane, we're pretty much done. Pick a point in the plane, say $(5, 0, 0)$, then the equation of the plane is

   $$\boxed{-6(x - 5) + 10(y - 0) - 15(z - 0) = 0} \quad \text{or} \quad \boxed{6x - 10y + 16z = 30}$$