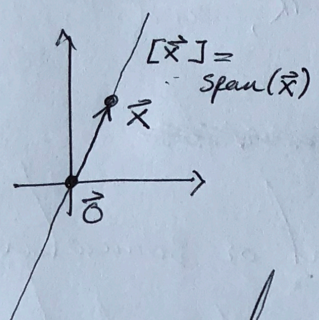
Projections

Let $\vec{x}$ and $\vec{y}$ be any nonzero vectors in $\mathbb{R}^n$, and let us consider whether and how we can ~~[illegible]~~ say that $\vec{x}$ is parallel to $\vec{y}$ or orthogonal or somewhere in between. This will then enable us to understand better the "in-" part of the Cauchy-Schwarz and triangle inequalities.

First, let us define parallelism between ~~two~~ vectors to mean "being scalar multiples of each other,"

$$\boxed{\text{"}\vec{x} \text{ is parallel to } \vec{y}\text{" means } \vec{x} = c\vec{y} \text{ for some } c \in \mathbb{R}}$$

If you want, you can prove that this defines an equivalence relation on $\mathbb{R}^n$"; and the equivalence class of a nonzero vector $\vec{x} \in \mathbb{R}^n$ is the line passing through the origin $\vec{0}$ and $\vec{x}$,



$$\boxed{\begin{aligned}[\vec{x}] &= \{\vec{y} \mid \vec{y} = c\vec{x}, \text{ for some } c \in \mathbb{R}\} \\ &= \{c\vec{x} \mid c \in \mathbb{R}\}\end{aligned}}$$

and is what, in chapter 4, we will call the span of $\vec{x}$. Thus, by our definition of parallel,

$$\boxed{\text{"}\vec{y} \text{ is parallel to } \vec{x}\text{" means } \vec{y} \in \operatorname{span}(\vec{x}) = [\vec{x}]}$$

So much for parallel. We will say $\vec{x}$ is orthogonal to $\vec{y}$ in $\mathbb{R}^n$ if

$$\boxed{\vec{x} \cdot \vec{y} = 0}$$

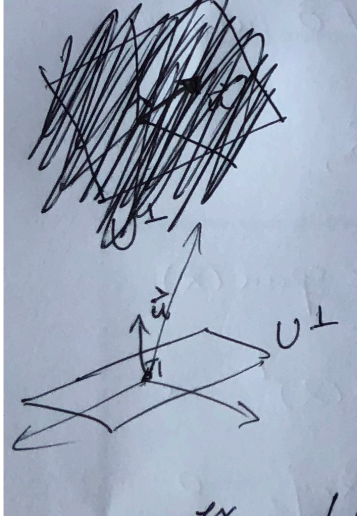
and write, in this case,

$$\boxed{\vec{x} \perp \vec{y}}$$

Let us also define the <u>orthogonal complement</u> of <u>a subset U of $\mathbb{R}^n$</u>:

$$\boxed{\begin{aligned} U^\perp &\equiv \underline{\text{orthogonal complement of } U \subseteq \mathbb{R}^n} \\ &:= \{\vec{v} \in \mathbb{R}^n \mid \vec{u} \perp \vec{v} \text{ for all } \vec{u} \in U\} \\ &= \{\vec{v} \in \mathbb{R}^n \mid \vec{u} \cdot \vec{v} = 0 \text{ for all } \vec{u} \in U\} \end{aligned}}$$

<u>ex.</u> Let $U = \{\vec{u}\}$ be a 1-element set, with $\vec{u} \neq \vec{0}$. Then, in $\mathbb{R}^3$ (in $\mathbb{R}^n, n \geq 4$, it's "<u>hyperplane</u>")



$$\boxed{\begin{aligned} U^\perp = \{\vec{u}\}^\perp = \ &\text{plane through } \vec{0} \\ &\text{\& orthogonal to } \vec{u} \\ &(\vec{u} \text{ is the "normal vector"} \\ &\text{to } U^\perp) \end{aligned}}$$

$\boxtimes$

<u>ex</u> $U = \text{span}(\vec{u}) \Rightarrow U^\perp = \{u\}^\perp =$ plane orthogonal to $\text{span}(\vec{u})$, & hence to $\vec{u}$.

<u>QED</u>

Returning to our two nonzero vectors $\vec{x}, \vec{y} \in \mathbb{R}^n$,

we ~~know~~ know that, by def.

(1) $\vec{x} \parallel \vec{y} \overset{\text{def}}{\iff} \vec{x} = c\vec{y}$ for some $c \in \mathbb{R}$ $\iff \vec{y} = k\vec{x}$ for some $k \in \mathbb{R}$

(2) $\vec{x} \perp \vec{y} \overset{\text{def}}{\iff} \vec{x} \cdot \vec{y} = 0$

What about any in-between cases?

Claim: We can always decompose $\vec{y} \neq \vec{0}$ into parallel and orthogonal components to $\vec{x}$,

$$\boxed{\vec{y} = \vec{y}_{\parallel} + \vec{y}_{\perp}, \quad \text{where } \vec{y}_{\parallel} = c\vec{x} \in \operatorname{span}(\vec{x}) \ \& \ \vec{y}_{\perp} \in \operatorname{span}(\vec{x})^{\perp}. \text{ i.e. } \vec{x} \cdot \vec{y}_{\perp} = 0}$$

namely $\boxed{\vec{y}_{\parallel} \equiv \operatorname{proj}_{\vec{x}} \vec{y} = \frac{\vec{x} \cdot \vec{y}}{\|\vec{x}\|^2} \vec{x}}$, $\boxed{\vec{y}_{\perp} := \vec{y} - \vec{y}_{\parallel}}$

pf: $\boxed{\vec{y}_{\parallel} = c\vec{x} \in \operatorname{span}(\vec{x})}$ by definition of parallel,

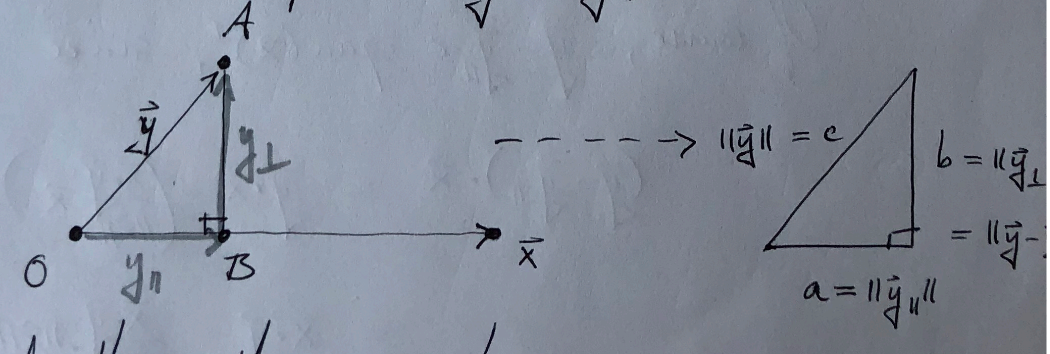
so what we need to show is that

$$\boxed{c = \frac{\vec{x}\cdot\vec{y}}{\|\vec{x}\|^2}}$$

for then will have

$$\boxed{\vec{y}_{\parallel} = c\vec{x} = \operatorname{proj}_{\vec{x}}\vec{y}}$$

Toward this end, consider the triangle ΔOAB with vertices O (the origin), A ("arrow tip" of $\vec{y}$) & B ("arrow tip" of $\vec{y}_{\parallel}$) in the plane $\operatorname{span}(\vec{x},\vec{y}) = \{a\vec{x}+b\vec{y} \mid a,b\in\mathbb{R}\}$ spanned by $\vec{x}$ & $\vec{y}$,



and apply the Pythagorean thm.:

$$a^2+b^2=c^2 \implies \|y_{\parallel}\|^2 + \|\vec{y}-\vec{y}_{\parallel}\|^2 = \|\vec{y}\|^2$$

Since $\vec{y}_{\parallel} = c\vec{x}$,

$$\Longrightarrow \quad \underbrace{c^2\|\vec{x}\|^2}_{=\|\vec{y}_{\parallel}\|^2} + \underbrace{\cancel{\|\vec{y}\|^2} - 2c\,\vec{x}\cdot\vec{y} + \overbrace{c^2\|\vec{x}\|^2}^{=\|\vec{y}_{\parallel}\|^2}}_{=\|\vec{y}-\vec{y}_{\parallel}\|^2}$$

$$= \|\vec{y}_{\parallel}\|^2 + \|\vec{y}-\vec{y}_{\parallel}\|^2 = \cancel{\|\vec{y}\|^2}$$

Pythag. Thm.

$$\Longrightarrow \quad \cancel{2}c^2\|\vec{x}\|^2 = \cancel{2}\cancel{c}\,\vec{x}\cdot\vec{y}$$

$$\Longrightarrow \quad c = \frac{\vec{x}\cdot\vec{y}}{\|\vec{x}\|^2}$$

Of course, $\vec{y}-\vec{y}_{\parallel} \in \operatorname{span}(\vec{x})^{\perp}$, since

$$(\vec{y}-\vec{y}_{\parallel})\cdot\vec{x} = \vec{y}\cdot\vec{x} - \left(\frac{\vec{x}\cdot\vec{y}}{\|\vec{x}\|^2}\vec{x}\right)\cdot\vec{x}$$

$$= \vec{x}\cdot\vec{y} - \frac{\vec{x}\cdot\vec{y}}{\cancel{\|\vec{x}\|^2}}\cancel{\|\vec{x}\|^2}$$

$$= 0$$

so $\vec{y}_{\perp} \perp \vec{x}$!

QED

ex. $\vec{x} = \langle 1,2,3\rangle$, $\vec{y} = \langle -2,1,-3\rangle \in \mathbb{R}^3 \setminus \{\vec{0}\}$:

$$\vec{x}_{\parallel} = \text{proj}_{\vec{y}}\vec{x} = \frac{\vec{x}\cdot\vec{y}}{\|\vec{y}\|^2}\vec{y}$$

$$= \frac{\langle 1,2,3\rangle \cdot \langle -2,1,-3\rangle}{\left(\sqrt{\langle -2,1,-3\rangle\cdot\langle -2,1,-3\rangle}\right)^2}\langle -2,1,-3\rangle$$

$$= \frac{\cancel{-2}+\cancel{2}-9}{4+1+9}\langle -2,1,-3\rangle$$

$$= \boxed{\frac{-9}{14}\langle -2,1,-3\rangle}$$

So

$$\vec{x}_{\perp} = \vec{x} - \vec{x}_{\parallel}$$

$$= \langle 1,2,3\rangle + \frac{9}{14}\langle -2,1,-3\rangle$$

$$= \left\langle \frac{14}{14} - \frac{18}{14}, \frac{28+9}{14}, \frac{42}{14} - \frac{27}{14}\right\rangle$$

$$= \left\langle -\frac{4}{14}, \frac{37}{14}, \frac{15}{14}\right\rangle$$

$$= \frac{1}{14}\langle -4, 37, 15\rangle$$

Then, note

$$\vec{x}_{\parallel} \cdot \vec{x}_{\perp} = \left(\frac{-9}{14}\langle -2, 1, -3\rangle\right) \cdot \left(\frac{1}{14}\langle -4, 37, 15\rangle\right)$$

$$= \frac{-9}{14^2}\cancelto{0}{\left(8 + 37 - 45\right)}$$

$$= 0$$

so $\vec{x}_{\parallel} \perp \vec{x}_{\perp}$. Moreover,

$$\vec{x}_{\parallel} + \vec{x}_{\perp} = \frac{-9}{14}\langle -2, 1, -3\rangle + \frac{1}{14}\langle -4, 37, 15\rangle$$

$$= \frac{1}{14}\left(\langle 18, -9, 27\rangle + \langle -4, 37, 15\rangle\right)$$

$$= \frac{1}{14}\langle 18-4, -9+37, 27+15\rangle$$

$$= \frac{1}{14}\langle 14, 28, 42\rangle$$

$$= \langle 1, 2, 3\rangle$$

$$= \vec{x}$$

QED

<u>Thm. 1.10</u> (<u>actually, a slight generalization</u>)

From the Cauchy-Schwarz inequality we know that for all vectors $\vec{x}, \vec{y} \in \mathbb{R}^n$,

$$|\vec{x} \cdot \vec{y}| \leq \|\vec{x}\| \|\vec{y}\|$$

When do we have equality? Our claim is that

$$\boxed{|\vec{x} \cdot \vec{y}| = \|\vec{x}\| \|\vec{y}\| \text{ iff } \vec{y} = c\vec{x} \text{ for some } c \in \mathbb{R}}$$

<u>pf:</u> Suppose first that $\vec{y} = c\vec{x}$ for some $c \in \mathbb{R}$, then

$$\begin{aligned} |\vec{x} \cdot \vec{y}| = |\vec{x} \cdot (c\vec{x})| &= |c| |\vec{x} \cdot \vec{x}| = |c| \|\vec{x}\|^2 \\ &= \|\vec{x}\| (|c| \|\vec{x}\|) \\ &= \|\vec{x}\| \|c\vec{x}\| \\ &= \|\vec{x}\| \|\vec{y}\| \end{aligned}$$

Suppose, conversely, that $|\vec{x} \cdot \vec{y}| = \|\vec{x}\| \|\vec{y}\|$, & consider

The projection $\vec{y}_{\parallel}$ of $\vec{y}$ onto $\vec{x}$,

$$\vec{y}_{\parallel} = \mathrm{proj}_{\vec{x}}\vec{y} = \frac{\vec{x}\cdot\vec{y}}{\|\vec{x}\|^2}\vec{x} \overset{\text{by assump.}}{=} \frac{\pm\cancel{\|\vec{x}\|}\|\vec{y}\|}{\|\vec{x}\|^{\cancel{2}}}\vec{x} = \pm\frac{\|\vec{y}\|}{\|\vec{x}\|}\vec{x} \qquad (*)$$

Then,

$$\begin{aligned}
\|\vec{y}_{\perp}\|^2 = \|\vec{y}-\vec{y}_{\parallel}\|^2 &= (\vec{y}-\vec{y}_{\parallel})\cdot(\vec{y}-\vec{y}_{\parallel}) \\
&= \vec{y}\cdot\vec{y} - \underbrace{2\vec{y}\cdot\vec{y}_{\parallel}}_{=2(\vec{y}_{\parallel}+\vec{y}_{\parallel})\cdot\vec{y}_{\parallel} = 2\vec{y}_{\parallel}\cdot\vec{y} \text{ since } \vec{y}_{\parallel}\cdot\vec{y}_{\perp}=0} + \vec{y}_{\parallel}\cdot\vec{y}_{\parallel} \\
&= \vec{y}\cdot\vec{y} - \vec{y}_{\parallel}\cdot\vec{y}_{\parallel} \\
&= \|\vec{y}\|^2 - \|\vec{y}_{\parallel}\|^2
\end{aligned}$$

Now we use our assumption that $|\vec{x}\cdot\vec{y}| = \|\vec{x}\|\|\vec{y}\|$, in the form $(*)$ at the top of this page: $\vec{y}_{\parallel} = \pm\frac{\|\vec{y}\|}{\|\vec{x}\|}\vec{x}$

$\Longrightarrow$ $\|\vec{y}_{\parallel}\| = \left\|\|\vec{y}\|\cdot\frac{\vec{x}}{\|\vec{x}\|}\right\| = \|\vec{y}\|\underbrace{\left\|\frac{\vec{x}}{\|\vec{x}\|}\right\|}_{=1} = \|\vec{y}\|$. Therefore,

$$\|\vec{y}_{\perp}\|^2 = \|\vec{y}\|^2 - \|\vec{y}_{\parallel}\|^2 = 0$$

$\Longrightarrow$ $\vec{y}_{\perp} = \vec{0}$, by Thm. 1.5.

QED

Cor. For all $\vec{x}, \vec{y} \in \mathbb{R}^n$,

$$\|\vec{x}+\vec{y}\| = \big|\|\vec{x}\| \pm \|\vec{y}\|\big| \iff \vec{y} = \vec{y}_{\parallel} \in \operatorname{span}(\vec{x})$$

pf: Suppose $\vec{y} = \vec{y}_{\parallel} = c\vec{x} \in \operatorname{span}(\vec{x})$. Then $|\vec{x}\cdot\vec{y}| = \|\vec{x}\|\|\vec{y}\|$ by Thm. 1.10, so (or equiv. $\vec{x}\cdot\vec{y} = \pm\|\vec{x}\|\|\vec{y}\|$)

$$\begin{aligned}
\|\vec{x}+\vec{y}\|^2 &= (\vec{x}+\vec{y})\cdot(\vec{x}+\vec{y}) \\
&= \vec{x}\cdot\vec{x} + 2\vec{x}\cdot\vec{y} + \vec{y}\cdot\vec{y} \\
&= \|\vec{x}\|^2 \pm 2\|\vec{x}\|\|\vec{y}\| + \|\vec{y}\|^2 \\
&= (\|\vec{x}\| \pm \|\vec{y}\|)^2
\end{aligned}$$

and therefore $\|\vec{x}+\vec{y}\| = \big|\|\vec{x}\| \pm \|\vec{y}\|\big|$.

Conversely, if $\|\vec{x}+\vec{y}\| = \big|\|\vec{x}\| \pm \|\vec{y}\|\big|$, then

$$\begin{aligned}
\cancel{\|\vec{x}\|^2} + 2\vec{x}\cdot\vec{y} + \cancel{\|\vec{y}\|^2} &= (\vec{x}+\vec{y})\cdot(\vec{x}+\vec{y}) \\
&= \|\vec{x}+\vec{y}\|^2 \\
&= (\|\vec{x}\| \pm \|\vec{y}\|)^2 \\
&= \cancel{\|\vec{x}\|^2} \pm 2\|\vec{x}\|\|\vec{y}\| + \cancel{\|\vec{y}\|^2}
\end{aligned}$$

QED

$$\implies \vec{x}\cdot\vec{y} = \pm\|\vec{x}\|\|\vec{y}\| \implies |\vec{x}\cdot\vec{y}| = \|\vec{x}\|\|\vec{y}\| \overset{\text{Thm. 1.10}}{\implies} \vec{y} = \vec{y}_{\parallel} \in \operatorname{span}(\vec{x}).$$

<u>Cor.</u> $\forall \vec{x}, \vec{y} \in \mathbb{R}^n$,

$$\boxed{\begin{aligned} &\|\vec{x}+\vec{y}\| = \|\vec{x}\| + \|\vec{y}\| \\ &\iff \vec{y} = \vec{y}_{\parallel} = c\vec{x} \in \operatorname{span}(\vec{x}) \\ &\qquad \text{for a } \underline{\text{positive}}\ c > 0. \end{aligned}}$$

<u>pf:</u> When $c = \frac{\vec{x}\cdot\vec{y}}{\|\vec{x}\|^2} > 0$, $\vec{x}\cdot\vec{y} > 0$, & the proof otherwise follows in the same way as that of the previous corollary. <u>QED</u>