Preliminaries for $\det(A^T) = \det A$

Prop. 0 (a) • $\boxed{A \in GL(n,\mathbb{R}) \iff A^T \in GL(n,\mathbb{R})}$,

and

$$\boxed{(A^T)^{-1} = (A^{-1})^T}$$

(b) $\boxed{A, B \in GL(n,\mathbb{R}) \iff AB \in GL(n,\mathbb{R})}$,

and

$$\boxed{(AB)^{-1} = B^{-1}A^{-1}}$$

pf: (a) $A \in GL(n,\mathbb{R}) \overset{\text{def}}{\iff} \exists B \in M_n(\mathbb{R})$ s.t. $AB = BA = I$, in which case $B \in GL(n,\mathbb{R})$ is unique & is denoted A^{-1}. Sim. w/ A^T. But since

$$(A^T)\underline{\underline{(A^T)^{-1}}} = I$$
$$= I^T$$
$$= (A^{-1}A)^T$$
$$= A^T\underline{\underline{(A^{-1})^T}}$$

(and similarly $(A^{-1})^T A^T = I$)

Strictly: $\exists A^{-1}$ but $\nexists (A^T)^{-1}$ implies a contradiction: $(A^{-1})^T = (A^T)^{-1}$ exists & doesn't exist. Etc.

{ we see that A^{-1} exists iff $(A^T)^{-1}$ does, in which case, by uniqueness of $(A^T)^{-1} = (A^{-1})^T$

(b) If $A, B \in GL(n,\mathbb{R})$, then

$$\begin{aligned}(AB)(B^{-1}A^{-1}) &= A(BB^{-1})A^{-1}\\ &= AA^{-1}\\ &= I \quad \text{therefore!}\\ &= (AB)(AB)^{-1}\end{aligned}$$

& sim. $(B^{-1}A^{-1})(AB) = I = (AB)^{-1}(AB)$,

so $AB \in GL(n,\mathbb{R})$, & in this case

$(AB)^{-1} = B^{-1}A^{-1}$!

If $(AB) \in GL(n,\mathbb{R})$, then $AB \sim I$, so

$\exists E_1, \dots, E_p \in GL(n,\mathbb{R})$ elementary matrices

s.t. $E_p \cdots E_1 = (AB)^{-1}$, i.e. $E_p \cdots E_1 (AB) = I$,

but by associativity

$$I = (E_p \cdots E_1)(AB) = (E_p \cdots E_1 A) B$$

which shows that $B^{-1} = E_p \cdots E_1 A$ exists,

& $\therefore$ $B \in GL(n,\mathbb{R})$. But then

Prop. 1 Any upper triangular square matrix

$$A = \begin{pmatrix} a_{11} & a_{12} & \cdots & a_{1n} \\ 0 & a_{22} & & \vdots \\ \vdots & \ddots & \ddots & \\ 0 & \cdots & 0 & a_{nn} \end{pmatrix} \text{ has}$$

$$\boxed{\det A = \prod_{i=1}^{n} a_{ii}}$$

pf: By induction: $n=1$, $\det(a) = a$ ✓

Now suppose true for $n \geq 1$ & consider $A \in M_{n+1}(\mathbb{R})$ upper triangular, $A = \begin{pmatrix} a_{11} & \cdots & a_{1,n+1} \\ & \ddots & \vdots \\ 0 & & a_{n+1,n+1} \end{pmatrix}$.

Expanding along the top row we have

$$\det A = \sum_{j=1}^{n+1} (-1)^{j+1} a_{1j} \det(A_{1j})$$

Claim: for all $j > 1$ we have $\det A_{1j} = 0$, because column 1 of each such A_{1j} is all 0's, $\vec{a}_{(1j),1} = \vec{0}$.

pf: By induction, we show that any $B \in M_n(\mathbb{R})$ w/ $\vec{b}_1 = \vec{0}$ has $\det B = 0$. For $n=1$,

$\det(0) = 0$

(Not informative enough? Consider $n=2$:
$B = \begin{pmatrix} 0 & b \\ 0 & d \end{pmatrix} \Rightarrow \det B = \det\begin{pmatrix} 0 & b \\ 0 & d \end{pmatrix} = 0 \cdot d - 0 \cdot b = 0.$)

Now suppose true for some $n \geq 1$ & consider $B \in M_{n+1}(\mathbb{R})$ with $\vec{b}_1 = \vec{0} \in \mathbb{R}^{n+1}$.

$$\det B = \det \begin{pmatrix} 0 & b_{12} & \cdots & b_{1,n+1} \\ 0 & b_{22} & \cdots & b_{2,n+1} \\ \vdots & \vdots & \ddots & \vdots \\ 0 & b_{n+1,2} & \cdots & b_{n+1,n+1} \end{pmatrix}$$

$$= \underbrace{0 \cdot \det B_{11}}_{=0} + \sum_{j=2}^{n} (-1)^{j+1} b_{1j} \underbrace{\det B_{1j}}_{\substack{=0 \text{ by the} \\ \text{induction hypothesis}}}$$

since each B_{1j}, for $j>1$, has 1st col. $\vec{0}$ (why?)

$= 0$

QED (of claim)

Thus, by this claim, we see that $\det A_{1j} = 0 \; \forall j > 1$,

so

$$\det A = (-1)^{1+1} a_{11} \underbrace{\det A_{11}}_{= \prod_{i=2}^{n+1} a_{ii} \text{ by induction hypothesis}} = \prod_{i=1}^{n} a_{ii}$$

QED

<u>Prop. 2</u> Any lower triangular matrix $A = \begin{pmatrix} a_{11} & 0 & \cdots & 0 \\ \vdots & \ddots & \ddots & \vdots \\ & & \ddots & 0 \\ a_{n1} & \cdots & \cdots & a_{nn} \end{pmatrix}$ $\in M_n(\mathbb{R})$ satisfies

$$\boxed{\det A = \prod_{i=1}^{n} a_{ii}}$$

<u>Pf:</u> By induction: $n=1$, $\det(a) = a$ ✓

Now suppose true for any $n \geq 1$ & consider $A \in M_{n+1}(\mathbb{R})$ upper triangular:

$$\det A = \det \begin{pmatrix} a_{11} & 0 & \cdots & \cdots & 0 \\ \vdots & & \ddots & & \vdots \\ \vdots & & & \ddots & 0 \\ a_{n+1,n} & \cdots & \cdots & \cdots & a_{n+1,n+1} \end{pmatrix}$$

$$= a_{11} \underbrace{\det A_{11}}_{\substack{= \prod_{i=2}^{n+1} a_{ii} \\ \text{by induction} \\ \text{hypothesis}}} + \underbrace{\sum_{j=2}^{n+1} (-1)^{j+1} \underset{\uparrow \atop = a_{1j},\ j>1}{0} \cdot \det A_{1j}}_{=0}$$

$$= \prod_{i=1}^{n+1} a_{ii}.$$

QED

$$I = E_p \cdots E_1 AB \implies B^{-1} = E_p \cdots E_1 A$$ (right-mult. both sides by B^{-1})

$$\implies \cancel{}\ I = (BE_p \cdots E_1)A$$

(left-mult. by B)

$$\implies A \in GL(n, \mathbb{R}) \ \& \ A^{-1} = BE_p \cdots E_1 .$$

In this case, too, we have $(AB)^{-1} = B^{-1}A^{-1}$ as above.

QED

$\det(A^T) = \det A$

Lemma 1 (3.9 (1)) For any elementary matrix E (representing either a type I, II or III row operation) we have

$$\boxed{\det(E^T) = \det E}$$

pf: (1) Since any type I (scaling a row) E is diagonal, $E = \begin{pmatrix} 1 & & 0 \\ & c & \\ 0 & & 1 \end{pmatrix}$, & so upper triangular,

$$\det E = c$$

But obviously E is symmetric, $E = E^T$, so trivially

$$\det(E^T) = \det E = c$$

(2) Similarly a type III matrix E, switching rows i & j, has rows $\underset{= \text{row } j \text{ of } I}{E_i = \vec{e}_j} = \langle 0, \dots, \overset{j\text{th slot}}{1}, \dots, 0 \rangle$, & $\underset{= \text{row } i \text{ of } I}{E_j = \vec{e}_i}$, so $E_{ij} = 1 = E_{ji}$. Everything else is as before, in the identity matrix, I, which

is diagonal & $\therefore$ symmetric. Thus, $E \in \mathrm{Sym}_n(\mathbb{R})$, & $E^T = E$, so $\det(E^T) = \det E$, trivially.

(3) Consider, then, a type II elementary matrix, adding k times row i of I to row j of I, $k\vec{e}_i + \vec{e}_j$: then $E_j = I + \begin{pmatrix} & 0 & \\ 0 \cdots & k & \cdots 0 \\ & 0 & \end{pmatrix}$

ijth position

Case 1: ~~…~~, $i > j$ e.g. $E = \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & k \\ 0 & 0 & 1 \end{pmatrix}$ $\overbrace{(k \cdot \text{row } 3 + \text{row } 2)}^{k\vec{e}_3 + \vec{e}_2}$

$i = 3,\ j = 2$

In this case E is upper triangular, so by Prop. 1 above $\det E = \prod_{i=1}^{n} E_{ii} = 1$, while by Prop. 2 E^T is lower triangular, so $\det E^T = \prod_{i=1}^{n} E_{ii} = 1$.

Case 2: $i < j$ – same, but E is lower triangular & E^T upper triangular, so

$\det E = 1 = \det E^T$

QED

Lemma 2 (3.9 (2)) $\forall A \in M_n(\mathbb{R})$,

$$\det(A^T) = \det A \implies \det((EA)^T) = \det(EA)$$

for any elementary matrix E.

Pf: Suppose $\det(A^T) = \det A$, & let us use Thm. 3.7 ($\det(AB) = (\det A)(\det B)$), Thm. 1.18 ($(AB)^T = B^T A^T$), & Lemma 1 above

$$\begin{aligned}
\det((EA)^T) &\overset{\text{Thm. 1.18}}{=} \det(A^T E^T) \\
&\overset{\text{Thm. 3.7}}{=} \det(A^T)\det(E^T) \\
&\overset{\text{assump. + Lemma 1}}{=} (\det A)\cdot(\det E) = (\det E)(\det A) \quad \text{bec. } \mathbb{R} \text{ is commut.} \\
&\overset{\text{Thm. 3.7}}{=} \det(EA)
\end{aligned}$$

QED

Exercise 1: We have seen that $\forall A, B \in M_n(\mathbb{R})$,

$$\boxed{\operatorname{tr}(AB) = \operatorname{tr}(BA)}$$

Show that we also have

$$\boxed{\det(AB) = \det(BA)}$$

Exercise 2: Show that $\forall A, B \in M_n(\mathbb{R})$, if $B \in GL(n, \mathbb{R})$, then

$$\operatorname{tr}(BAB^{-1}) = \operatorname{tr}(A)$$

and

$$\det(BAB^{-1}) = \det A$$

Cor. (Lemma 3.9 (3)) For any sequence $E_1, \dots, E_p$ of elementary matrices, we have

$$\det(E_1 \cdots E_p) = \det(E_p \cdots E_1)$$
$$= \det((E_p \cdots E_1)^T) = \det((E_1 \cdots E_p)^T)$$

pf: By induction, using Ex. 1 & Thm 3.7 + Lemmas [illegible] QED

Thm. 3.10 $\boxed{\det(A^T) = \det A,}$ $\forall A \in M_n(\mathbb{R})$.

pf: Case 1, $A \in GL(n,\mathbb{R})$: $A \in GL(n,\mathbb{R}) \iff$ $A \sim I$ (row equiv.), so $E_p \cdots E_1 A = I$ for a sequence $E_i \in GL(n,\mathbb{R})$ of (invertible) elementary matrices. Writing $F_i = E_i^{-1} \in GL(n,\mathbb{R})$, we have

$$E_p \cdots E_1 A = I \iff E_{p-1} \cdots E_1 A = E_p^{-1}$$
$$= F_p$$
$$\iff \cdots \iff \boxed{A = F_1 \cdots F_p}$$

we have, by the last Corollary,

$$\det(A^T) = \det((F_1 \cdots F_p)^T)$$
$$= \det(F_1 \cdots F_p)$$
$$= \det A$$

Case 2, $A \notin GL(n,\mathbb{R})$ By Prop. 0, $A \notin GL(n,\mathbb{R}) \iff$ $A^T \notin GL(n,\mathbb{R})$, so by Thm. 3.5 $\det(A^T) = 0 = \det A$.

QED

Thm. 3.11 (Cofactor Expansion Along Any Row or Column)

Let

$$A = \begin{pmatrix} a_{11} & \cdots & a_{1n} \\ \vdots & \ddots & \vdots \\ a_{n1} & \cdots & a_{nn} \end{pmatrix} = \begin{pmatrix} -\vec{A}_1- \\ \vdots \\ -\vec{A}_n- \end{pmatrix} = \begin{pmatrix} | & & | \\ \vec{a}_1 & \cdots & \vec{a}_n \\ | & & | \end{pmatrix}$$

Then, $\det A$ may be computed by cofactor expansion along any row $i \in \{1, \dots, n\}$ & any column $j \in \{1, \dots, n\}$, as

$$\det A \overset{\text{def}}{=} \sum_{j=1}^{n} (-1)^{1+j} a_{1j} \det A_{1j} \quad \begin{pmatrix}\text{expanding} \\ \text{along 1st row}\end{pmatrix}$$

$$= \sum_{j=1}^{n} (-1)^{i+j} a_{ij} \det A_{ij} \quad \begin{pmatrix}\text{expanding} \\ \text{along the } i\text{th} \\ \text{row}\end{pmatrix}$$

$$= \sum_{i=1}^{n} (-1)^{i+j} a_{ij} \det A_{ij} \quad \begin{pmatrix}\text{expanding} \\ \text{along } j\text{th column}\end{pmatrix}$$

Terminology:

$$\underbrace{(-1)^{i+j} a_{ij} \underbrace{\det \underbrace{A_{ij}}_{\substack{ij \text{ submatrix} \\ \text{(delete row } i\text{, col } j \\ \text{of } A)}}}_{ij \text{ minor}}}$$

$$\underbrace{(-1)^{i+j} \det A_{ij}}_{ij \text{ cofactor}}$$

Pf: Suppose $i > 1$, & notice that if E is the row-swap $\vec{A}_1 \leftrightarrow \vec{A}_i$, then $\det(EA)$ has its top row $\vec{A}_i$, so by Thm. 3-

$$-\det A = \det(EA) = \det\begin{pmatrix}\vec{A}_i\\ \vec{A}_2\\ \vdots\\ \vec{A}_1\\ \vdots\\ \vec{A}_n\end{pmatrix} \leftarrow i\text{th row}$$

$$= \sum_{j=1}^{n} (-1)^{j+1} a_{ij} \det(EA)_{1j}$$

Now, it takes $i-2$ rows swaps to get

$$\begin{pmatrix}\vec{A}_2\\ \vdots\\ \vec{A}_{i-1}\\ \vec{A}_1\\ \vdots\\ \vec{A}_n\end{pmatrix} \overset{①}{\longmapsto} \begin{pmatrix}\vec{A}_2\\ \vdots\\ \vec{A}_{i-2}\\ \vec{A}_1\\ \vec{A}_{i-1}\\ \vdots\\ \vec{A}_n\end{pmatrix} \overset{②}{\longmapsto} \cdots \overset{i-2}{\longmapsto} \begin{pmatrix}\vec{A}_1\\ \vec{A}_2\\ \vdots\\ \vec{A}_n\end{pmatrix}$$

each contributing a (-1) to $\det(EA)_{1j}$, so letting $E_1, \ldots, E_{i-2}$ be these $(n-1)\times(n-1)$ row swaps, we get

$$-\det A = \sum_{j=1}^{n} (-1)^{j+1} a_{ij} \det(EA)_{1j}$$

$$= \sum_{j=1}^{n} \underbrace{(-1)^{j+1+(i-2)}}_{=(-1)^{j+i-1}} a_{ij} \det A_{ij}$$

$\Rightarrow$ (mult. both sides by (-1)) $\det A = \sum_{j=1}^{n} (-1)^{i+j} a_{ij} \det A_{ij}$.

proving the 1st.

For the second equality, we use Thm. 3.10, $\det A = \det A^T$, and the above equality:

$$\det A = \det A^T$$

$$= \sum_{j=1}^{n} (-1)^{i+j} \underbrace{(A^T)_{ij}}_{=a_{ji}} \underbrace{\det (A^T)_{ij}}_{=A_{ji}}$$

$$= \sum_{j=1}^{n} (-1)^{i+j} a_{ji} \det A_{ji}$$

QED