Further Considerations About $\mathbb{R}^n$

<u>Thm. 1.3</u> (cf. Exercise 22 in Sec. 1.1) $\mathbb{R}^n$ is a real vector space (in our abstract sense, cf. 4.1) under elementwise addition & scalar multiplication: $\forall \vec{x}, \vec{y}, \vec{z} \in \mathbb{R}^n$, $\forall c, d \in \mathbb{R}$,

$(\mathbb{R}^n, +)$ is an abelian group (a Lie group, in fact)

(1) $\boxed{\vec{x}+\vec{y} = \vec{y}+\vec{x}}$ (+ is commutative)

(2) $\boxed{\vec{x}+(\vec{y}+\vec{z}) = (\vec{x}+\vec{y})+\vec{z}}$ (+ is associative)

(3) $\exists \vec{0} := \langle 0, \ldots, 0 \rangle \in \mathbb{R}^n$ s.t. $\boxed{\vec{x}+\vec{0} = \vec{0}+\vec{x} = \vec{x}}$

(4) $\forall \vec{x} \in \mathbb{R}^n$, $\exists (-\vec{x}) \in \mathbb{R}^n$ s.t. $\boxed{\vec{x}+(-\vec{x}) = (-\vec{x})+\vec{x} = \vec{0}}$

namely, if $\vec{x} = \langle x_1, \ldots, x_n \rangle$, then

$\boxed{-\vec{x} := \langle -x_1, \ldots, -x_n \rangle = -1 \cdot \langle x_1, \ldots, x_n \rangle.}$

"Compatibility" of scalar mult. w/ vector addit. ($\mathbb{R}$ acts on $\mathbb{R}^n$)

(5) $\boxed{c(\vec{x}+\vec{y}) = c\vec{x}+c\vec{y}}$ (distribution over + in $\mathbb{R}^n$)

(6) $\boxed{(c+d)\vec{x} = c\vec{x}+d\vec{x}}$ (distribution over + in $\mathbb{R}$)

(7) $\boxed{(cd)\vec{x} = c(d\vec{x})}$ (associativity of scalar multiplic.)

(8) $\boxed{1 \cdot \vec{x} = \vec{x}}$ ($1 \in \mathbb{R}$ here)

The proof is merely a tedious verification of the eight conditions, which hold because we defined both operations ($+$ and $\cdot_s$ (scalar mult)) <u>elementwise</u>, and elementwise we are just dealing with real numbers under $+$ and $\cdot$ on $\mathbb{R}$. But let us ~~illustrate~~ this on, say, the proofs of (1) and (7):

<u>pf of (1)</u>: Let $\vec{x} = \langle x_1, \ldots, x_n \rangle \in \mathbb{R}^n$, $\vec{y} = \langle y_1, \ldots, y_n \rangle \in \mathbb{R}^n$.
Then,

$$\vec{x} + \vec{y} = \langle x_1, \ldots, x_n \rangle + \langle y_1, \ldots, y_n \rangle$$

$$= \langle x_1 + y_1, \ldots, x_n + y_n \rangle \quad \text{by def of } + \text{ in } \mathbb{R}^n$$

$$= \langle y_1 + x_1, \ldots, y_n + x_n \rangle \quad \text{bec. } + \text{ in } \mathbb{R} \text{ is commut.}$$

$$= \langle y_1, \ldots, y_n \rangle + \langle x_1, \ldots, x_n \rangle \quad \text{by def of } + \text{ in } \mathbb{R}^n$$

$$= \vec{y} + \vec{x}$$

<u>QED</u>

<u>pf of (7)</u>: Let $c, d \in \mathbb{R}$ and $\vec{x} = \langle x_1, \ldots, x_n \rangle \in \mathbb{R}^n$.

Then,

$$
\begin{aligned}
(cd)\vec{x} &= (cd)\langle x_1, \ldots, x_n \rangle \\
&= \langle (cd)x_1, \ldots, (cd)x_n \rangle && \text{by def. of } \cdot_s \\
&= \langle c(dx_1), \ldots, c(dx_n) \rangle && \cdot \text{ is assoc. in } \mathbb{R} \\
&= c\langle dx_1, \ldots, dx_n \rangle && \text{by def. of } \cdot_s \\
&= c\,(d\langle x_1, \ldots, x_n \rangle) && \text{by def. of } \cdot_s \\
&= c(d\vec{x})
\end{aligned}
$$

<u>QED</u>

The other six statements follow similarly, and have accordingly been left to the exercises (#22 in Sec. 1.1).

Thm. 1.5 The dot product is a symmetric, positive definite bilinear form on $\mathbb{R}^n$: $\forall \vec{x}, \vec{y}, \vec{z} \in \mathbb{R}^n$; $\forall c \in \mathbb{R}$

(1) $\boxed{\vec{x}\cdot\vec{y} = \vec{y}\cdot\vec{x}}$ (symmetry)

(2) $\boxed{\vec{x}\cdot\vec{x} = \|\vec{x}\|^2 \geq 0,}$ and

(3) $\boxed{\|\vec{x}\| = 0 \iff \vec{x}\cdot\vec{x} = 0 \iff \vec{x} = \vec{0}}$

(positive definiteness)

the dot prod. is a bilinear form on $\mathbb{R}^n$

(4) $\boxed{\underbrace{c}_{\in\mathbb{R}}\,(\underbrace{\vec{x}\cdot\vec{y}}_{\in\mathbb{R}}) = \underbrace{(\underbrace{c\vec{x}}_{\in\mathbb{R}^n})\cdot\underbrace{\vec{y}}_{\in\mathbb{R}^n}}_{\in\mathbb{R}} = \underbrace{\underbrace{\vec{x}}_{\in\mathbb{R}^n}\cdot(\underbrace{c\vec{y}}_{\in\mathbb{R}^n})}_{\in\mathbb{R}}}$

(scalar mult in 2nd variable)

product in $\mathbb{R}$

(scalar mult. in 1st variable)

(5) $\boxed{\vec{x}\cdot(\vec{y}+\vec{z}) = \vec{x}\cdot\vec{y} + \vec{x}\cdot\vec{z}}$ (additivity in the second variable)

(6) $\boxed{(\vec{x}+\vec{y})\cdot\vec{z} = \vec{x}\cdot\vec{z} + \vec{y}\cdot\vec{z}}$ (additivity in first variable)

Remark: Linearity in the 1st variable of the dot product means

(a) $\boxed{c(\vec{x}\cdot\vec{y}) = (c\vec{x})\cdot\vec{y}}$ $\forall c \in \mathbb{R}$, $\forall \vec{x} \in \mathbb{R}^n$; keeping $\vec{y} \in \mathbb{R}^n$ fixed

(b) $\boxed{(\vec{x}+\vec{y})\cdot\vec{z} = \vec{x}\cdot\vec{z} + \vec{y}\cdot\vec{z}}$ $\forall \vec{x}, \vec{y} \in \mathbb{R}^n$, keeping $\vec{z} \in \mathbb{R}^n$ fixed

Similarly, linearity in the 2nd variable of the dot product is, by definition, the following conditions:

(a) $\boxed{c(\vec{x}\cdot\vec{y}) = \vec{x}\cdot(c\vec{y})}$ $\forall \vec{y}\in\mathbb{R}^n$, $\forall c\in\mathbb{R}$, keeping $\vec{x}\in\mathbb{R}^n$ fixed

(b) $\boxed{\vec{x}\cdot(\vec{y}+\vec{z}) = \vec{x}\cdot\vec{y} + \vec{x}\cdot\vec{z}}$, $\forall \vec{y},\vec{z}\in\mathbb{R}^n$, keeping $\vec{x}\in\mathbb{R}^n$ fixed

It is in this sense that conditions (4)–(6) show that the dot product is bilinear. The "form" part in "bilinear form" simply means that the output values of the dot product are real numbers, not vectors. (for example, the cross product is also bilinear, but not a form — it is, rather, a vector-valued (outputting vectors in $\mathbb{R}^3$) bilinear function. ⊠

Now to the proof: again, a tedious exercise in combining the definition of $\vec{x}\cdot\vec{y}$ w/ the algebraic properties of $\mathbb{R}$. Let us prove (2) & (3) to illustrate.

pf of (2): Let $\vec{x} = \langle x_1, \ldots, x_n \rangle \in \mathbb{R}^n$; Then, since by def. $\|\vec{x}\| = \sqrt{\vec{x}\cdot\vec{x}}$, we have

$$\vec{x}\cdot\vec{x} = \|\vec{x}\|^2 = \sum_{i=1}^{n} x_i^2$$

When $n=1$, this means $\vec{x}\cdot\vec{x} = x_1^2 = x_1 x_1$, a product of a real number, x_1, with itself, so (cf. Chapter 1 of Rudin's "Principles of Mathematical Analysis")

$$\vec{x}\cdot\vec{x} = x_1^2 \geq 0$$

Now use induction on n: suppose $\vec{x}\cdot\vec{x} \geq 0$ for all $\vec{x} \in \mathbb{R}^n$; & consider $\vec{x} \in \mathbb{R}^{n+1}$, i.e. $\vec{x} = \langle x_1, \ldots, x_n, x_{n+1} \rangle$. Now, let $\vec{x}\,' = \langle x_1, \ldots, x_n \rangle$ be the "projection" of $\vec{x}$ onto $\mathbb{R}^n$, the 1st n

components (e.g. if $\vec{x} = \langle x, y, z \rangle \in \mathbb{R}^3$, $\vec{x}' = \langle x, y \rangle \in \mathbb{R}^2$).

Then,

$$\vec{x} \cdot \vec{x} = \sum_{i=1}^{n+1} x_i^2$$

$$= \underbrace{\left(\sum_{i=1}^{n} x_i^2 \right)}_{\geq 0} + \underbrace{x_{n+1}^2}_{\geq 0}$$

≥ 0 by induction hypothesis, bec. $\sum_{i=1}^{n} x_i^2 = \|\vec{x}'\|^2$

≥ 0 by properties of $\mathbb{R}$

≥ 0 bec. the sum of two nonneg. real #s is nonneg. (cf. Rudin)

QED

Pf (3): First, for any real # a, $a^2 = 0 \iff a = 0$ (Rudin), so automatically $\|\vec{x}\| = 0 \iff \|x\|^2 = 0$
$\iff \vec{x} \cdot \vec{x} = 0$.

We already know that $\vec{x} \cdot \vec{x} = \|x\|^2 \geq 0$, by (2), so let us suppose $\vec{x} \cdot \vec{x} = 0$. By induction on n, we will prove $\vec{x} = \vec{0}$:

For $n=1$, the result follows bec. $a^2=0 \iff a=0$:

$\vec{x} = \langle x_1 \rangle \implies$.

$$0 = \vec{x} \cdot \vec{x} = x_1^2 \iff x_1 = 0$$

$$\iff \vec{x} = \langle 0 \rangle = \vec{0}$$

Suppose, then, that for some $n \geq 1$ we have $\vec{x} \cdot \vec{x} = 0 \iff \vec{x} = \vec{0}$, and consider $\vec{x} \in \mathbb{R}^{n+1}$. Again, if $\vec{x} = \langle x_1, \ldots, x_n, x_{n+1} \rangle$, we write $\vec{x}' = \langle x_1, \ldots, x_n \rangle$, and note that

$$0 = \vec{x} \cdot \vec{x} = \sum_{i=1}^{n+1} x_i^2$$

$$= \left(\sum_{i=1}^{n} x_i^2 \right) + x_{n+1}^2$$

$$= \|\vec{x}'\|^2 + x_{n+1}^2$$

We already know that $\|\vec{x}'\|^2 \geq 0$ (by (2)) & $x_{n+1}^2 \geq 0$ (Rudin), and that if the sum of two nonneg. real numbers $c, d \geq 0$ is 0, $c+d=0$, then $c=d=0$, else say if $c>0$, then $0<c=-d\leq$

Thus, $\|\vec{x}'\|^2 = x_{n+1}^2 = 0$, & this $\Leftrightarrow$ $\|\vec{x}'\| = x_{n+1} = 0$ (Rudin), & therefore by the induction hypothesis $\Leftrightarrow \vec{x}' = \vec{0} \in \mathbb{R}^n$. Since $x_{n+1} = 0$, too, we see that $\|\vec{x}\|^2 = \vec{0} \Leftrightarrow \vec{x} = \langle \underbrace{x_1, \ldots, x_n}_{=\vec{0}=\vec{x}'}, \underbrace{x_{n+1}}_{=0} \rangle = \vec{0}$. QED

Remark: The fact that $\|\vec{x}\| = 0 \Leftrightarrow \vec{x} = \vec{0}$ assures us that for any nonzero vector $\vec{x} \in \mathbb{R}^n \setminus \{\vec{0}\}$ we have $\|\vec{x}\| > 0$. This allows us to scale $\vec{x}$ by $\frac{1}{\|\vec{x}\|}$, and in this way derive a unit vector "pointing in the same direction as $\vec{x}$." (i.e. a scalar multiple of x):

$$\boxed{\begin{aligned} \left\| \frac{1}{\|\vec{x}\|}\vec{x} \right\| &= \frac{1}{\|\vec{x}\|}\|\vec{x}\| \\ &= 1 \end{aligned}}$$

bc.
$$\begin{aligned} \|c\vec{x}\| &= \sqrt{(c\vec{x})\cdot(c\vec{x})} \\ &= \sqrt{c^2(\vec{x}\cdot\vec{x})} \\ &= |c|\sqrt{\vec{x}\cdot\vec{x}} \\ &= |c|\|\vec{x}\|, \ \forall c \in \mathbb{R} \end{aligned}$$

The procedure of scaling a nonzero vector into a unit-length scalar multiple of itself is called **normalizing** the vector $\vec{x} \in \mathbb{R}^n \setminus \{\vec{0}\}$. ☒

Def. (**unit n-sphere**) Let us define the **unit n-sphere** in $\mathbb{R}^{n+1}$ to be just the collection of unit-length vectors:

$$\boxed{S^n := \{ \vec{x} \in \mathbb{R}^{n+1} \mid \|\vec{x}\| = 1 \}}$$

This is just the collection of points a distance of 1 from the origin $\vec{0}$. ☒

Normalizing may be understood as a function

$$\boxed{f : \mathbb{R}^{n+1} \setminus \{\vec{0}\} \longrightarrow S^n, \qquad f(\vec{x}) := \frac{\vec{x}}{\|\vec{x}\|}}$$

<u>Lemma 1.6</u> For all unit vectors $\vec{x}, \vec{y} \in S^{n-1} \subseteq \mathbb{R}^n$, we have

$$\boxed{|\vec{x} \cdot \vec{y}| \leq 1} \qquad (1)$$

or equivalently

$$\boxed{-1 \leq \vec{x} \cdot \vec{y} \leq 1} \qquad (2)$$

<u>Pf:</u> <u>Step 1</u>:
$$\begin{aligned} 0 &\leq \|\vec{x}+\vec{y}\|^2 \quad (\text{Thm. } 1.5\ (2)) \\ &= (\vec{x}+\vec{y}) \cdot (\vec{x}+\vec{y}) \quad (\text{def.}) \\ &= \vec{x}\cdot\vec{x} + 2\vec{x}\cdot\vec{y} + \vec{y}\cdot\vec{y} \quad (\text{Thm. } 1.5) \\ &= \|\vec{x}\|^2 + 2\vec{x}\cdot\vec{y} + \|\vec{y}\|^2 \\ &\leq 2 + 2\vec{x}\cdot\vec{y} \quad (\text{since } \vec{x}, \vec{y} \in S^{n-1}) \\ &= 2(1+\vec{x}\cdot\vec{y}) \end{aligned}$$

$$\Longrightarrow \quad 0 \leq 1 + \vec{x}\cdot\vec{y} \quad \Longrightarrow \quad \boxed{-1 \leq \vec{x}\cdot\vec{y}}$$

which is half of (2). Step 2 is the other half:

<u>Step 2</u>:
$$\begin{aligned} 0 &\leq \|\vec{x}-\vec{y}\|^2 \\ &= (\vec{x}-\vec{y}) \cdot (\vec{x}-\vec{y}) \\ &= \vec{x}\cdot\vec{x} - 2\vec{x}\cdot\vec{y} + \vec{y}\cdot\vec{y} \\ &= \|\vec{x}\|^2 - 2\vec{x}\cdot\vec{y} + \|\vec{y}\|^2 \\ &\leq 2 - 2\vec{x}\cdot\vec{y} \quad \Longrightarrow \quad \boxed{\vec{x}\cdot\vec{y} \leq 1.} \end{aligned}$$

QED

Thm. (Cauchy - Schwarz) For all $\vec{x}, \vec{y} \in \mathbb{R}^n$,

$$|\vec{x} \cdot \vec{y}| \leq \|\vec{x}\| \|\vec{y}\|$$

pf: Case 1 (Either $\vec{x}$ or $\vec{y}$, or both, are $\vec{0}$)

If we suppose even one of $\vec{x}$ or $\vec{y}$ are $\vec{0}$, say $\vec{x} = \vec{0}$, then $\vec{x} \cdot \vec{y} = \vec{0} \cdot \vec{y} = \sum_{i=1}^{n} 0 y_i = 0$, so $|\vec{x} \cdot \vec{y}| = 0$. Similarly, by Thm. 1.5, $\vec{x} = \vec{0} \iff \|\vec{x}\| = 0$, so we trivially have

$$|\vec{x} \cdot \vec{y}| = 0 = 0 \cdot \|\vec{y}\| = \|\vec{x}\| \|\vec{y}\|$$

Case 2 ($\vec{x}, \vec{y} \neq \vec{0}$) If $\vec{x}, \vec{y} \in \mathbb{R}^n \setminus \{\vec{0}\}$, then we normalize them, $\frac{\vec{x}}{\|\vec{x}\|}, \frac{\vec{y}}{\|\vec{y}\|} \in S^{n-1}$, & apply the lemma & Thm. 1.5:

$$\left| \left(\frac{\vec{x}}{\|\vec{x}\|}\right) \cdot \left(\frac{\vec{y}}{\|\vec{y}\|}\right) \right| \leq 1$$

$$\implies \frac{1}{\|\vec{x}\| \|\vec{y}\|} |\vec{x} \cdot \vec{y}| \leq 1 \implies |\vec{x} \cdot \vec{y}| \leq \|\vec{x}\| \|\vec{y}\|.$$

QED

<u>Alternative proof</u>: Let $c = \frac{\vec{x}\cdot\vec{y}}{\|\vec{y}\|^2}$ and note that

$$
\begin{aligned}
0 &\leq \|\vec{x} - c\vec{y}\|^2 \qquad (\text{Thm. } 1.5\ (2)) \\
&= (\vec{x} - c\vec{y})\cdot(\vec{x} - c\vec{y}) \\
&= \vec{x}\cdot\vec{x} - 2c\,\vec{x}\cdot\vec{y} + c^2\,\vec{y}\cdot\vec{y} \\
&= \|\vec{x}\|^2 - 2\left(\frac{\vec{x}\cdot\vec{y}}{\|\vec{y}\|^2}\right)(\vec{x}\cdot\vec{y}) + \left(\frac{\vec{x}\cdot\vec{y}}{\|\vec{y}\|^2}\right)^2\|\vec{y}\|^2 \\
&= \|\vec{x}\|^2 \underbrace{- 2\frac{(\vec{x}\cdot\vec{y})^2}{\|\vec{y}\|^2} + \frac{(\vec{x}\cdot\vec{y})^2}{\|\vec{y}\|^{\cancel{4}\,2}}\cancel{\|\vec{y}\|^2}}_{= -\frac{\vec{x}\cdot\vec{y}}{\|\vec{y}\|^2}} \\
&= \|\vec{x}\|^2 - \frac{(\vec{x}\cdot\vec{y})^2}{\|\vec{y}\|^2}
\end{aligned}
$$

$$
\Rightarrow \quad \frac{(\vec{x}\cdot\vec{y})^2}{\|\vec{y}\|^2} \leq \|x\|^2 \Rightarrow (\vec{x}\cdot\vec{y})^2 \leq \|\vec{x}\|^2\|\vec{y}\|^2
$$

$$
\Rightarrow |\vec{x}\cdot\vec{y}| \leq \|\vec{x}\|\|\vec{y}\| \qquad \underline{\text{QED}}
$$

Thm. 1.8 (Triangle Inequality) For all $\vec{x}, \vec{y} \in \mathbb{R}^n$,

$$\boxed{\|\vec{x}+\vec{y}\| \leq \|\vec{x}\| + \|\vec{y}\|}$$

pf:
$$\|\vec{x}+\vec{y}\|^2 = (\vec{x}+\vec{y})\cdot(\vec{x}+\vec{y})$$
$$= \underbrace{\vec{x}\cdot\vec{x}}_{=\|\vec{x}\|^2} + \underbrace{2\vec{x}\cdot\vec{y}}_{\leq 2\|\vec{x}\|\|\vec{y}\|} + \underbrace{\vec{y}\cdot\vec{y}}_{=\|\vec{y}\|^2}$$
$$\leq \|\vec{x}\|^2 + 2\|\vec{x}\|\|\vec{y}\| + \|\vec{y}\|^2$$ (by Cauchy-Schwarz)
$$= (\|\vec{x}\|+\|\vec{y}\|)^2$$

$$\Longrightarrow \|\vec{x}+\vec{y}\| \leq \|\vec{x}\|+\|\vec{y}\|.$$ QED

We can use Cauchy-Schwarz for another thing, too, to define angle measures between vectors $\vec{x}, \vec{y} \in \mathbb{R}^n$ for $n \geq 4$, by using cosine:

$$\boxed{\theta := \cos^{-1}\left(\frac{\vec{x}\cdot\vec{y}}{\|\vec{x}\|\|\vec{y}\|}\right)}$$

since range (cos) = $[-1, 1]$, which is exactly the range of possible ~~values~~ of $\frac{\vec{x}\cdot\vec{y}}{\|\vec{x}\|\|\vec{y}\|}$!