Detailed Proof of the 4x4 Case of a Row-Swap: $\det(EA) = -\det A$

4x4: switch rows 1 & 2:

$$\det(EA) = \det\begin{pmatrix} a_{21} & a_{22} & a_{23} & a_{24} \\ a_{11} & a_{12} & a_{13} & a_{14} \\ a_{31} & a_{32} & a_{33} & a_{34} \\ a_{41} & a_{42} & a_{43} & a_{44} \end{pmatrix}$$

$$= a_{21}\det\begin{pmatrix} a_{12} & a_{13} & a_{14} \\ a_{32} & a_{33} & a_{34} \\ a_{42} & a_{43} & a_{44} \end{pmatrix} - a_{22}\det\begin{pmatrix} a_{11} & a_{13} & a_{14} \\ a_{31} & a_{33} & a_{34} \\ a_{41} & a_{43} & a_{44} \end{pmatrix}$$

$$+ a_{23}\det\begin{pmatrix} a_{11} & a_{12} & a_{14} \\ a_{31} & a_{32} & a_{34} \\ a_{41} & a_{42} & a_{44} \end{pmatrix} - a_{24}\det\begin{pmatrix} a_{11} & a_{12} & a_{13} \\ a_{31} & a_{32} & a_{33} \\ a_{41} & a_{42} & a_{43} \end{pmatrix}$$

$$= \underbrace{a_{21}\Big(a_{12}}_{\text{II } 1}\det\begin{pmatrix} a_{33} & a_{34} \\ a_{43} & a_{44} \end{pmatrix} - \overset{\text{III } 1}{a_{13}}\det\begin{pmatrix} a_{32} & a_{34} \\ a_{42} & a_{44} \end{pmatrix} + \underbrace{a_{14}}_{\text{IV } 1}\det\begin{pmatrix} a_{32} & a_{33} \\ a_{42} & a_{43} \end{pmatrix}\Big)$$

$$- \underbrace{a_{22}\Big(a_{11}}_{\text{I } 1}\det\begin{pmatrix} a_{33} & a_{34} \\ a_{43} & a_{44} \end{pmatrix} - \overset{\text{III } 2}{a_{13}}\det\begin{pmatrix} a_{31} & a_{34} \\ a_{41} & a_{44} \end{pmatrix} + \underbrace{a_{14}}_{\text{IV } 2}\det\begin{pmatrix} a_{31} & a_{33} \\ a_{41} & a_{43} \end{pmatrix}\Big)$$

$$+ \underbrace{a_{23}\Big(a_{11}}_{\text{I } 2}\det\begin{pmatrix} a_{32} & a_{34} \\ a_{42} & a_{44} \end{pmatrix} - \underbrace{a_{12}}_{\text{II } 2}\det\begin{pmatrix} a_{31} & a_{34} \\ a_{41} & a_{44} \end{pmatrix} + \underbrace{a_{14}}_{\text{IV } 3}\det\begin{pmatrix} a_{31} & a_{32} \\ a_{41} & a_{42} \end{pmatrix}\Big)$$

$$- \underbrace{a_{24}\Big(a_{11}}_{\text{I } 3}\det\begin{pmatrix} a_{32} & a_{33} \\ a_{42} & a_{43} \end{pmatrix} - \underbrace{a_{12}}_{\text{II } 3}\det\begin{pmatrix} a_{31} & a_{33} \\ a_{41} & a_{43} \end{pmatrix} + \overset{\text{III } 3}{a_{13}} \cdot \det\begin{pmatrix} a_{31} & a_{32} \\ a_{41} & a_{42} \end{pmatrix}\Big)$$

(collect everything w/ a_{11} in it, then $a_{12}, \dots, a_{14}$

(i.e. regroup in terms of det A, hopefully))

$$= -\overset{\boxed{\mathrm{I}}}{\boxed{a_{11}}}\left(\overset{\textcircled{1}}{\boxed{a_{22}}}\det\begin{pmatrix}a_{33} & a_{34}\\ a_{43} & a_{44}\end{pmatrix} - \overset{\textcircled{2}}{\boxed{a_{23}}}\det\begin{pmatrix}a_{32} & a_{34}\\ a_{42} & a_{44}\end{pmatrix} + \overset{\textcircled{3}}{\boxed{a_{24}}}\det\begin{pmatrix}a_{32} & a_{33}\\ a_{42} & a_{43}\end{pmatrix}\right)$$

$$+\overset{\boxed{\mathrm{II}}}{\boxed{a_{12}}}\left(\overset{\textcircled{1}}{\boxed{a_{21}}}\det\begin{pmatrix}a_{33} & a_{34}\\ a_{43} & a_{44}\end{pmatrix} - \overset{\textcircled{2}}{\boxed{a_{23}}}\det\begin{pmatrix}a_{31} & a_{34}\\ a_{41} & a_{44}\end{pmatrix} + \overset{\textcircled{3}}{\boxed{a_{24}}}\det\begin{pmatrix}a_{31} & a_{33}\\ a_{41} & a_{43}\end{pmatrix}\right)$$

$$-\overset{\boxed{\mathrm{III}}}{\boxed{a_{13}}}\left(\overset{\textcircled{1}}{\boxed{a_{21}}}\det\begin{pmatrix}a_{32} & a_{34}\\ a_{42} & a_{44}\end{pmatrix} - \overset{\textcircled{2}}{\boxed{a_{22}}}\det\begin{pmatrix}a_{31} & a_{34}\\ a_{41} & a_{44}\end{pmatrix} + \overset{\textcircled{3}}{\boxed{a_{24}}}\det\begin{pmatrix}a_{31} & a_{32}\\ a_{41} & a_{42}\end{pmatrix}\right)$$

$$+\overset{\boxed{\mathrm{IV}}}{\boxed{a_{14}}}\left(\overset{\textcircled{1}}{\boxed{a_{21}}}\det\begin{pmatrix}a_{32} & a_{33}\\ a_{42} & a_{43}\end{pmatrix} - \overset{\textcircled{2}}{\boxed{a_{22}}}\det\begin{pmatrix}a_{31} & a_{33}\\ a_{41} & a_{43}\end{pmatrix} + \overset{\textcircled{3}}{\boxed{a_{23}}}\det\begin{pmatrix}a_{31} & a_{32}\\ a_{41} & a_{42}\end{pmatrix}\right)$$

$$= -\overset{\boxed{\mathrm{I}}}{\boxed{a_{11}}}\det\begin{pmatrix}\boxed{a_{22}} & \boxed{a_{23}} & \boxed{a_{24}}\\ a_{32} & a_{33} & a_{34}\\ a_{42} & a_{43} & a_{44}\end{pmatrix} + \overset{\boxed{\mathrm{II}}}{\boxed{a_{12}}}\det\begin{pmatrix}\boxed{a_{21}} & \boxed{a_{23}} & \boxed{a_{24}}\\ a_{31} & a_{33} & a_{34}\\ a_{41} & a_{43} & a_{44}\end{pmatrix}$$

$$-\overset{\boxed{\mathrm{III}}}{\boxed{a_{13}}}\det\begin{pmatrix}\boxed{a_{21}} & \boxed{a_{22}} & \boxed{a_{24}}\\ a_{31} & a_{32} & a_{34}\\ a_{41} & a_{42} & a_{44}\end{pmatrix} + \overset{\boxed{\mathrm{IV}}}{\boxed{a_{14}}}\det\begin{pmatrix}\boxed{a_{21}} & \boxed{a_{22}} & \boxed{a_{23}}\\ a_{31} & a_{32} & a_{33}\\ a_{41} & a_{42} & a_{43}\end{pmatrix}$$

$$= -\det\begin{pmatrix}\boxed{a_{11}} & \boxed{a_{12}} & \boxed{a_{13}} & \boxed{a_{14}}\\ \boxed{a_{21}} & \boxed{a_{22}} & \boxed{a_{23}} & \boxed{a_{24}}\\ a_{31} & a_{32} & a_{33} & a_{34}\\ a_{41} & a_{42} & a_{43} & a_{44}\end{pmatrix} = -\det A$$

Let us write

$$D_{3,4} = \det\begin{pmatrix} a_{33} & a_{34} \\ a_{43} & a_{44} \end{pmatrix}$$

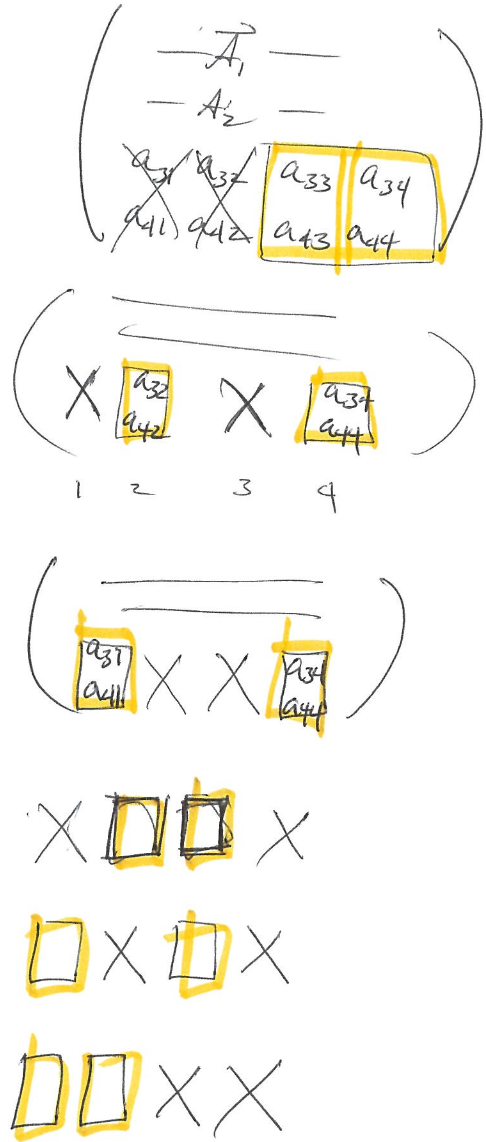
$$D_{2,4} = \det\begin{pmatrix} a_{32} & a_{34} \\ a_{42} & a_{44} \end{pmatrix}$$

$$D_{1,4} = \det\begin{pmatrix} a_{31} & a_{34} \\ a_{41} & a_{44} \end{pmatrix}$$

$$D_{2,3} = \det\begin{pmatrix} a_{32} & a_{33} \\ a_{42} & a_{43} \end{pmatrix}$$

$$D_{1,3} = \det\begin{pmatrix} a_{31} & a_{33} \\ a_{41} & a_{43} \end{pmatrix}$$

$$D_{1,2} = \det\begin{pmatrix} a_{31} & a_{32} \\ a_{41} & a_{42} \end{pmatrix}$$



Then the above eqns become

$$\begin{aligned}\det(EA) = {} & a_{21}\left(a_{12} D_{34} - a_{13} D_{24} + a_{14} D_{23}\right) \\ & - a_{22}\left(a_{11} D_{34} - a_{13} D_{14} + a_{14} D_{13}\right) \\ & + a_{23}\left(a_{11} D_{24} - a_{12} D_{14} + a_{14} D_{12}\right) \\ & - a_{24}\left(a_{11} D_{23} - a_{12} D_{13} + a_{13} D_{12}\right)\end{aligned}$$

(a_{2j}'s here)

$$= -a_{11}\left(a_{22}D_{34} - a_{23}D_{24} + a_{24}D_{23}\right)$$

$$+a_{12}\left(a_{21}D_{34} - a_{23}D_{14} + a_{24}D_{13}\right) \qquad (a_{ik} \text{ here})$$

$$-a_{13}\left(a_{21}D_{24} - a_{22}D_{14} + a_{24}D_{12}\right)$$

$$+a_{14}\left(a_{21}D_{23} - a_{22}D_{13} + a_{23}D_{12}\right)$$

$$= -\det A$$

i.e. $\qquad$ (j, k, r, s distinct elements of $\{1,2,3,4\}$)

$$\det(EA) = \sum_{j=1}^{4} (-1)^{j+1} a_{2j} \left(\sum_{j<k} (-1)^{k} a_{1k} D_{rs} + \sum_{j>k} (-1)^{k+1} a_{1k} D_{rs} \right)$$

$$= -\sum_{k=1}^{4} (-1)^{k+1} a_{1k} \left(\sum_{j>k} (-1)^{j+1} a_{2j} D_{rs} + \sum_{j<k} (-1)^{j} a_{2j} D_{rs} \right)$$

$$= -\det A$$