Points and Vectors

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1 Points and Vectors

1.1 Definitions

We denote n**-dimensional Euclidean space** by

$$\boxed{\begin{aligned}\mathbb{R}^n &= \overbrace{\mathbb{R}\times\cdots\times\mathbb{R}}^{n \text{ times}} \\ &= \{(x_1,\ldots,x_n) \mid x_i \text{ is a real number}\}\end{aligned}} \tag{1.1}$$

We will typically use capital letters P for the elements of $\mathbb{R}^n$, the n-tuples $(x_1,\ldots,x_n)$, which we call **points**,

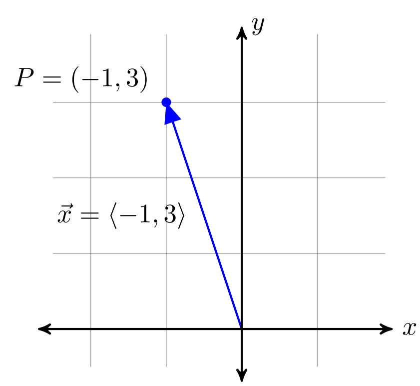
$$\boxed{P = (x_1,\ldots,x_n)} \tag{1.2}$$

We may also think of the elements of $\mathbb{R}^n$ as **vectors**, however, for example when working with things like velocities/forces/etc. ('vector quantities', typically residing in phase space rather than configuration space, in physics lingo). There is a certain notation which is found in both physics and math books, that emphasises the *vector* part of the elements of $\mathbb{R}^n$, and it is the arrow and angle bracket notations for the n-tuples:

$$\boxed{\vec{x} \;\text{ or }\; \mathbf{x} = \langle x_1,\ldots,x_n\rangle} \tag{1.3}$$

Example 1.1 *Our two main examples are the Euclidean plane* $\mathbb{R}^2$ *and Euclidean three-dimensional space* $\mathbb{R}^3$*. We usually denote* x_1 *by* x*,* x_2 *by* y*, and* x_3 *by* z*, so that* $P=(x,y)$ *or* (x,y,z)*, as the case may be.* ■

We visualize points and vectors differently. For example, in $\mathbb{R}^2$ the point $P=(-1,3)$ is pictured as a dot, and the vector $\vec{x}=\langle -1,3\rangle$ is pictured as an arrow from the origin to P:



1.2 Algebraic Properties of Vectors

The essential difference between points and vectors, mathematically, is that points don't possess any algebraic properties, whereas vectors do. The key algebraic properties of vectors are *addition*, *scalar multiplication*, and the *dot* and *cross products*:

(1) We can add two vectors.

(2) We can scale any vector (multiply it by a real number).

(3) Rules (1) and (2) are subject to certain rules (associativity, commutativity and distributivity rules) which will make them "nice" to work with and give them geometric content.[1]

(4) We can "multiply" two vectors, in different ways. The dot product of two vectors results in a real number.

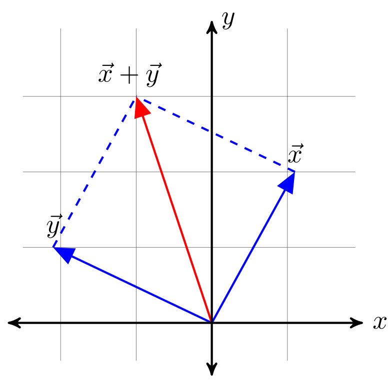
(5) The cross product results in another vector.

1.2.1 Vector Addition

We define **addition** of vectors in $\mathbb{R}^n$ componentwise,

$$\boxed{\begin{aligned}\vec{x}+\vec{y} &= \langle x_1, x_2, \ldots, x_n\rangle + \langle y_1, y_2, \ldots, y_n\rangle \\ &= \langle x_1+y_1,\ x_2+y_2, \ldots,\ x_n+y_n\rangle\end{aligned}} \tag{1.4}$$

Example 1.2 *Let us see what this means in $\mathbb{R}^2$. Take, say, $\vec{x} = \langle 1, 2\rangle$ and $\vec{y} = \langle -2, 1\rangle$. Then $\vec{x}+\vec{y} = \langle 1-2, 2+1\rangle = \langle -1, 3\rangle$.*



Thus we see that to reach $\vec{x}+\vec{y}$, we may first go to $\vec{x}$, then go in the direction of $\vec{y}$ to get to $\vec{x}+\vec{y}$, or else we may to to $\vec{y}$ first and then go in the direction of $\vec{x}$. This shows geometrically the algebraic rule called <u>commutativity</u>, *$\vec{x}+\vec{y} = \vec{y}+\vec{x}$.* ■

As a direct consequence of our definition of vector addition see that we have **commutativity**

[1] In the case of Euclidean $\mathbb{R}^n$ the rules for addition and scalar multiplication follow from directly from the *definitions* of addition and scalar multiplication themselves, but in abstract linear algebra they form the basis for the definition of (abstract) vector space.

of addition:

$$\begin{aligned}\vec{x}+\vec{y} &= \langle x_1,\ldots,x_n\rangle+\langle y_1,\ldots,y_n\rangle \\ &= \langle x_1+y_1,\ldots,x_n+y_n\rangle \\ &= \langle y_1+x_1,\ldots,y_n+x_n\rangle \\ &= \vec{y}+\vec{x}\end{aligned} \tag{1.5}$$

Another obvious fact about our definition of addition in $\mathbb{R}^n$ is that it is **associative**. We can add two vectors first, then a third, but we could just as well have added the second and third first, then the first to the result:

$$(\vec{x}+\vec{y})+\vec{z}=\vec{x}+(\vec{y}+\vec{z}) \tag{1.6}$$

which again follows from the same associativity holding in each component, $(x_i+y_i)+z_i = x_i+(y_i+z_i)$.

It is also clear that the **zero** vector,

$$\boxed{\vec{0}=\langle 0,\ldots,0\rangle} \tag{1.7}$$

satisfies

$$\vec{0}+\vec{x}=\vec{x}+\vec{0}=\vec{x} \tag{1.8}$$

for all $\vec{x}$ in $\mathbb{R}^n$. Moreover, the **negative** or a vector $\vec{x}$ in $\mathbb{R}^n$, defined by

$$-\vec{x}=\langle -x_1,\ldots,-x_n\rangle \tag{1.9}$$

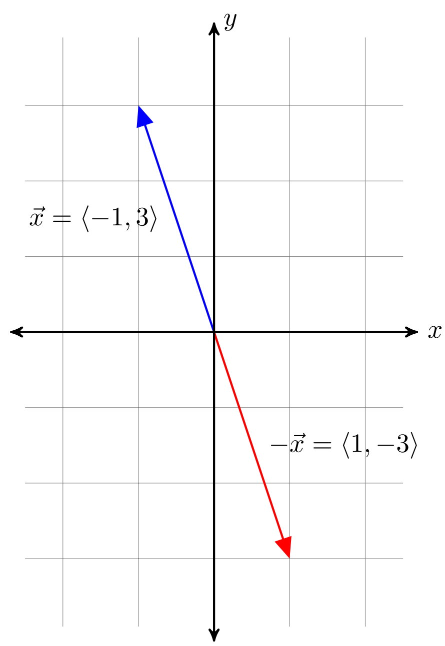
satisfies

$$(-\vec{x})+\vec{x}=\vec{x}+(-\vec{x})=\vec{0}$$

which we may more compactly write $-\vec{x}+\vec{x}=\vec{x}-\vec{x}=\vec{0}$. That is, we may use negative vectors to define **subtraction** of vectors in $\mathbb{R}^n$, namely by addition of negatives:

$$\vec{x}-\vec{y}=\vec{x}+(-\vec{y}) \tag{1.10}$$

Geometrically, the negative $-\vec{x}$ of a vector is the reflection of $\vec{x}$ through the origin:



1.2.2 Scalar Multiplication

Let us now define **scalar multiplication** of vectors in $\mathbb{R}^n$, meaning multiplication of a vector $\vec{x}$ by a real number a. As with addition, this is defined componentwise:

$$\boxed{a\vec{x} = a\langle x_1, \ldots, x_n\rangle = \langle ax_1, \ldots, ax_n\rangle} \tag{1.11}$$

It is clear that

$$\begin{aligned} 1\vec{x} &= \vec{x} & (1.12) \\ 0\vec{x} &= \vec{0} & (1.13) \end{aligned}$$

Moreover, we have **associativity of scalar multiplication**,

$$a(b\vec{x}) = (ab)\vec{x} \tag{1.14}$$

and **distributivity** of scalar multiplication over addition:

$$\begin{aligned} a(\vec{x}+\vec{y}) &= a\vec{x} + a\vec{y} & (1.15) \\ (a+b)\vec{x} &= a\vec{x} + b\vec{x} & (1.16) \end{aligned}$$

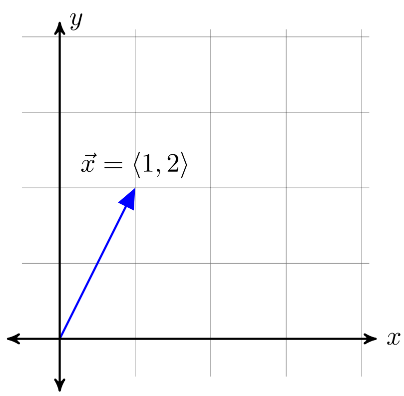
for all real numbers a and b and all elements $\vec{x}$ and $\vec{y}$ of $\mathbb{R}^n$.

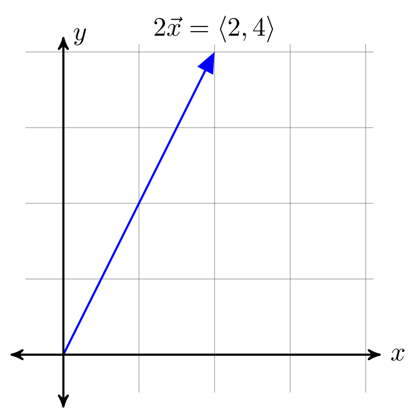
The geometric content of scalar multiplication may be seen in the following example:

Example 1.3 *Take, say, the vector* $\vec{x} = \langle 1, 2\rangle$ *and the real number* $a = 2$. *Then, algebraically,*

$$a\vec{x} = 2\langle 1, 2\rangle = \langle 2\cdot 1,\ 2\cdot 2\rangle = \langle 2, 4\rangle$$

which, geometrically means this:





Thus geometrically scalar multiplication has the effect of scaling the length *of the vector* $\vec{x}$. ■

1.2.3 Coordinate Vectors and Vector Decomposition

We can use addition and scalar multiplicaton of vectors to decompose a given vector $\vec{x} = \langle x_1, \ldots, x_n \rangle$ into its **components** x_i:

$$\begin{aligned}\vec{x} &= \langle x_1, \ldots, x_n \rangle \\ &= \langle x_1, 0, \ldots, 0\rangle + \langle 0, x_2, 0, \ldots, 0\rangle + \cdots + \langle 0, \ldots, 0, x_n\rangle \\ &= x_1\langle 1, 0, \ldots, 0\rangle + x_2\langle 0, 1, 0, \ldots, 0\rangle + \cdots + x_n\langle 0, \ldots, 0\rangle\end{aligned}$$

If we define the **coordinate basis vectors**

$$\boxed{\mathbf{e}_1 = \langle 1, 0, \ldots, 0\rangle, \quad \mathbf{e}_2 = \langle 0, 1, 0 \ldots, 0\rangle, \quad \ldots, \quad \mathbf{e}_n = \langle 0, \ldots, 0, 1\rangle} \tag{1.17}$$

then we can nicely rewrite equation (1.17) as

$$\boxed{\vec{x} = \langle x_1, \ldots, x_n\rangle = x_1\mathbf{e}_1 + x_2\mathbf{e}_2 + \cdots + x_n\mathbf{e}_n} \tag{1.18}$$

Example 1.4 *When $n = 2$, we have special notation, which is more commonly found in physics texts:*

$$\boxed{\begin{aligned}\vec{i} &= \mathbf{i} = \mathbf{e}_1 = \langle 1, 0\rangle \\ \vec{j} &= \mathbf{j} = \mathbf{e}_2 = \langle 0, 1\rangle\end{aligned}} \tag{1.19}$$

When $n = 3$, we also write:

$$\boxed{\begin{aligned}\vec{i} &= \mathbf{i} = \mathbf{e}_1 = \langle 1, 0, 0\rangle \\ \vec{j} &= \mathbf{j} = \mathbf{e}_2 = \langle 0, 1, 0\rangle \\ \vec{k} &= \mathbf{k} = \mathbf{e}_3 = \langle 0, 0, 1\rangle\end{aligned}} \tag{1.20}$$

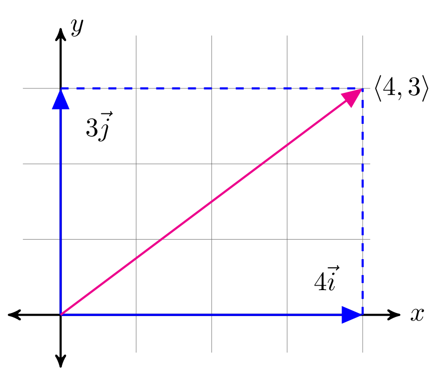
For example,

$$\langle 1, 3, -2\rangle = 1\vec{i} + 3\vec{j} - 2\vec{k} \tag{1.21}$$

is decomposed into its components. As another example,

$$\langle 4, 3\rangle = 4\vec{i} + 3\vec{j} \tag{1.22}$$

and this can be pictured as follows:



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1.2.4 The Relationship Between Points and Vectors: the Displacement Vector

Given two points $P = (x_1, \ldots, x_n)$ and $Q = (y_1, \ldots, y_n)$ in $\mathbb{R}^n$, the **displacement vector** from P to Q is defined as

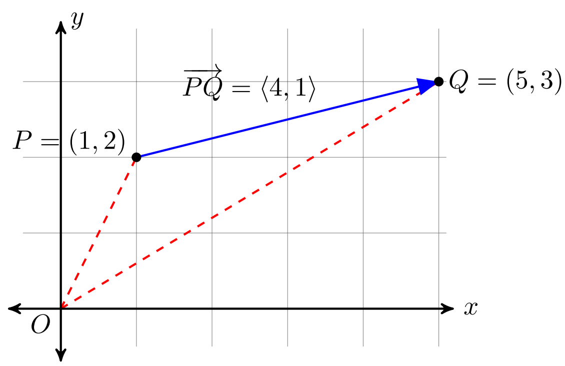
$$\boxed{\begin{aligned} \overrightarrow{PQ} &= \overrightarrow{OQ} - \overrightarrow{OP} \\ &= \langle y_1, \ldots, y_n \rangle - \langle x_1, \ldots, x_n \rangle \\ &= \langle y_1 - x_1, \ldots, y_n - x_n \rangle \end{aligned}} \tag{1.23}$$

We *picture* the displacement vector $\overrightarrow{PQ}$ as emanating from the point P and ending in an arrow at the point Q (even though it always, strictly speaking, emanates from the origin to its endpoint).

Example 1.5 *Let us look at the case of $\mathbb{R}^2$. Suppose $P = (1, 2)$ and $Q = (5, 3)$. Then,*

$$\overrightarrow{PQ} = \overrightarrow{OQ} - \overrightarrow{OP} = \langle 5, 3 \rangle - \langle 1, 2 \rangle = \langle 5 - 1,\ 3 - 2 \rangle = \langle 4, 1 \rangle \tag{1.24}$$

and the picture is this:



Remark 1.6 *In fact, $\overrightarrow{PQ}$ should be pictured as emanating from the origin, but we* want *to think of $\overrightarrow{PQ}$ as emanating from P. We should, then, if we were being rigorous, think of $\overrightarrow{PQ}$ as lying in a copy of $\mathbb{R}^n$ sitting above our position space $\mathbb{R}^n$ at the point P, that is we should think of $\overrightarrow{PQ}$ as lying in the set $\{P\} \times \mathbb{R}^n = \{(P, \vec{x}) \mid \vec{x} = \langle x_1, \ldots, x_n \rangle\}$, and thus*

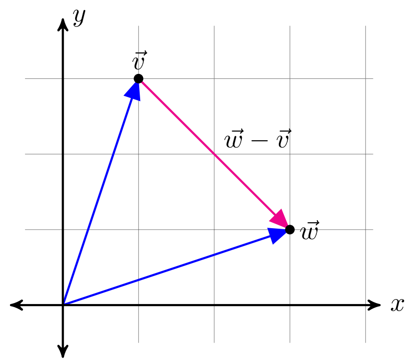
$$\overrightarrow{PQ} = (P, \overrightarrow{OQ} - \overrightarrow{OP})$$

For example, if $P = (1, 2)$ and $Q = (5, 3)$, then

$$\overrightarrow{PQ} = \Big((1, 2),\ \langle 4, 1 \rangle \Big)$$

We will not nit-pick here, and we will simply conflate points and vectors in the strict sense, but we will picture *vectors as emanating from points in the underlying position space.* ■

Now suppose we are considering not two points P and Q, but two vectors $\vec{v}$ and $\vec{w}$. Then we can consider the **displacement vector** from $\vec{v}$ to $\vec{w}$. This is, in fact $\vec{w} - \vec{v}$, which is simply due to the fact that $\vec{w} = \vec{v} + (\vec{w} - \vec{v})$:



This of course harmonizes with our previous definition, $\overrightarrow{PQ} = \overrightarrow{OQ} - \overrightarrow{OP}$. We subtract our starting position vector from our ending position vector in both cases.

1.2.5 The Dot Product

The dot product of two vectors $\vec{x} = \langle x_1, x_2, \ldots, x_n \rangle$ and $\vec{y} = \langle y_1, y_2, \ldots, y_n \rangle$ in $\mathbb{R}^n$ is defined by

$$\boxed{\vec{x} \cdot \vec{y} \;=\; \langle x_1, x_2, \ldots, x_n \rangle \cdot \langle y_1, y_2, \ldots, y_n \rangle \;=\; x_1 y_1 + x_2 y_2 + \cdots + x_n y_n} \tag{1.25}$$

or, more concisely, using summation notation,

$$\boxed{\vec{x} \cdot \vec{y} = \sum_{i=1}^{n} x_i y_i} \tag{1.26}$$

Example 1.7 *Consider the vectors $\vec{x} = \langle 1, 2, -5 \rangle$ and $\vec{y} = \langle -3, 2, 4 \rangle$ in $\mathbb{R}^3$. Their dot product is*

$$\begin{aligned} \vec{x} \cdot \vec{y} &= \langle 1, 2, -5 \rangle \cdot \langle -3, 2, 4 \rangle \\ &= 1 \cdot (-3) + 2 \cdot 2 + (-5) \cdot 4 \\ &= -19 \end{aligned}$$

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The **length** (or **magnitude** or **norm**) of a vector $\vec{x}$ in $\mathbb{R}^n$ will be defined as the square root of the dot product of $\vec{x}$ with itself:

$$\boxed{|\vec{x}| \;=\; \sqrt{\vec{x} \cdot \vec{x}}} \tag{1.27}$$

Thus,

$$|\vec{x}| = \sqrt{\sum_{i=1}^{n} x_i^2}$$

and therefore

$$|\vec{x}|^2 = \sum_{i=1}^{n} x_i^2$$

This relates to the formula for the **distance between two points** $P = (x_1, \ldots, x_n)$ and $Q = (y_1, \ldots, y_n)$:

$$\boxed{d(P,Q) \;=\; d\Big((x_1, \ldots, x_n),\ (y_1, \ldots, y_n)\Big) \;=\; \sqrt{\sum_{i=1}^{n}(y_i - x_i)^2}} \tag{1.28}$$

In the cases $n = 2$ and $n = 3$, this is the Pythagorean theorem. For $n > 3$ it's simply a definition which in a way takes the Pythagorean theorem as an axiom.

What is the relationship between the distance formula and the length of a vector? It is that the length of the displacement vector $\overrightarrow{PQ}$ is precisely the distance between P and Q:

$$\begin{aligned}
|\overrightarrow{PQ}| &= \sqrt{\overrightarrow{PQ} \cdot \overrightarrow{PQ}} \\
&= \sqrt{(\vec{Q} - \vec{P}) \cdot (\vec{Q} - \vec{P})} \\
&= \sqrt{\langle y_1 - x_1, \ldots, y_n - x_n \rangle \cdot \langle y_1 - x_1, \ldots, y_n - x_n \rangle} \\
&= \sqrt{\sum_{i=1}^{n}(y_i - x_i)^2} \\
&= d(P,Q)
\end{aligned}$$

The proofs of the following propertis of the dot product are analogous to the above calculation showing that $|\overrightarrow{PQ}| = d(P,Q)$, and we leave them as an easy exercise.

Example 1.8 *Let $P = (1,2)$ and $Q = (5,3)$ be points in the plane $\mathbb{R}^2$. Find the magnitude of the displacement vector from P to Q.*

Solution: Let $\vec{x} = \overrightarrow{PQ} = \overrightarrow{OQ} - \overrightarrow{OP} = \langle 4, 1 \rangle$, then

$$|\vec{x}| = |\langle 4, 1 \rangle| = \sqrt{4^2 + 1^2} = \sqrt{17}$$

■

Proposition 1.9 (Algebraic Properties of the Dot Product) *Let $\vec{x}$, $\vec{y}$, $\vec{z}$ be vectors in $\mathbb{R}^n$ and let c be a real number. Then,*

(1) $\vec{x} \cdot \vec{x} = |\vec{x}|^2$

(2) $|c\vec{x}| = |c||\vec{x}|$

(3) $\vec{x} \cdot \vec{y} = \vec{y} \cdot \vec{x}$ *(commutativity)*

(4) $\vec{x} \cdot (\vec{y} + \vec{z}) = \vec{x} \cdot \vec{y} + \vec{x} \cdot \vec{z}$ *(distributivity over addition)*

(5) $(c\vec{x}) \cdot \vec{y} = c(\vec{x} \cdot \vec{y}) = \vec{x} \cdot (c\vec{y})$ *(associativity and commutativity of scalar multiplication and dot multiplication)*

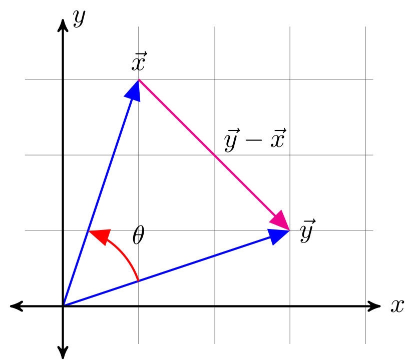
(6) $\vec{0} \cdot \vec{x} = 0$

In two and three dimensions, the dot product has a very *geometric* interpretation:

Proposition 1.10 *Let $\vec{x} = \langle x_1, y_1, z_1 \rangle$ and let $\vec{y} = \langle x_2, y_2, z_2 \rangle$ be two vectors in $\mathbb{R}^3$, and let θ be the angle between them. Then,*

$$\boxed{\vec{x} \cdot \vec{y} = |\vec{x}||\vec{y}| \cos\theta} \tag{1.29}$$

Proof: Consider the triangle formed by the vectors $\vec{x}$, $\vec{y}$ and $\vec{y} - \vec{x}$.



By the Law of Cosines

$$|\vec{y} - \vec{x}|^2 = |\vec{x}|^2 + |\vec{y}|^2 - 2|\vec{x}||\vec{y}| \cos\theta$$

Using the fact that $|\vec{x}|^2 = \vec{x} \cdot \vec{x}$, we can rewrite the above as

$$(\vec{y} - \vec{x}) \cdot (\vec{y} - \vec{x}) = \vec{x} \cdot \vec{x} + \vec{y} \cdot \vec{y} - 2|\vec{x}||\vec{y}| \cos\theta$$

Distributing on the left and simplifying, we get

$$\cancel{\vec{y} \cdot \vec{y}} - 2\vec{x} \cdot \vec{y} + \cancel{\vec{x} \cdot \vec{x}} = \cancel{\vec{x} \cdot \vec{x}} + \cancel{\vec{y} \cdot \vec{y}} - 2|\vec{x}||\vec{y}| \cos\theta$$

That is,

$$\cancel{-2}\vec{x} \cdot \vec{y} = \cancel{-2}|\vec{x}||\vec{y}| \cos\theta \qquad \blacksquare$$

1.2.6 Cross Product

Note that this section requires knowledge of determinants, which is to be found below, in the section on matrices.

The cross product is a *vector* product, meaning that multiplying two vectors this way results in another vector (this is why the dot product is sometimes called the *scalar* product, to distinguish it from this vector product). The cross product is only defined in 3 dimensions, i.e. only on $\mathbb{R}^3$.[2]

[2] It generalizes to other dimensions only once we switch to the **wedge product** (or **exterior product**) in multilinear algebra. Moving past the algebra we are led to a *differential* type of wedge product in the apparatus of **differential forms**. For further reading on this, see Knapp [2] and Gallier [1] for the algebra, and Munkres [3] for the calculus side.

Let $\vec{u} = \langle a, b, c\rangle$ and $\vec{v} = \langle d, e, f\rangle$ be two vectors in $\mathbb{R}^3$. We define their **cross product** to be the vector gotten by computing the following determinant:

$$\begin{aligned} \vec{u}\times\vec{v} \;=\; \langle a,b,c\rangle \times \langle d,e,f\rangle \;&=\; \begin{vmatrix} \vec{i} & \vec{j} & \vec{k} \\ a & b & c \\ d & e & f \end{vmatrix} \\ &=\; \vec{i}\begin{vmatrix} b & c \\ e & f \end{vmatrix} - \vec{j}\begin{vmatrix} a & c \\ d & f \end{vmatrix} + \vec{k}\begin{vmatrix} a & b \\ d & e \end{vmatrix} \end{aligned} \tag{1.30}$$

Carrying out this computation to its awful end, we get

$$\vec{u}\times\vec{v} \;=\; (bf-ce)\vec{i} - (af-cd)\vec{j} + (ae-bd)\vec{k} \;=\; \langle bf-ce,\ cd-af,\ ae-bd\rangle$$

But it is easier to remember the equation (1.30) in terms of determinants and just perform the rest of the computation by hand in particular cases.

Example 1.11 *Let $\vec{u} = \langle 1, 2, -2\rangle$ and $\vec{v} = \langle -8, 5, 4\rangle$. Then,*

$$\begin{aligned} \vec{u}\times\vec{v} &= \langle 1,2,-2\rangle \times \langle -8,5,4\rangle \\ &= \begin{vmatrix} \vec{i} & \vec{j} & \vec{k} \\ 1 & 2 & -2 \\ -8 & 5 & 4 \end{vmatrix} \\ &= \vec{i}\begin{vmatrix} 2 & -2 \\ 5 & 4 \end{vmatrix} - \vec{k}\begin{vmatrix} 1 & -2 \\ -8 & 4 \end{vmatrix} + \vec{j}\begin{vmatrix} 1 & 2 \\ -8 & 5 \end{vmatrix} \\ &= (8+10)\vec{i} - (4-16)\vec{j} + (5+16)\vec{k} \\ &= \boxed{18\vec{i} + 12\vec{j} + 21\vec{k}} \quad \textit{or} \quad \boxed{\langle 18, 12, 21\rangle} \end{aligned}$$

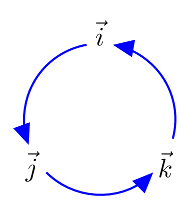
Example 1.12 *Let us compute $\vec{i}\times\vec{j}$:*

$$\vec{i}\times\vec{j} = \begin{vmatrix} \vec{i} & \vec{j} & \vec{k} \\ 1 & 0 & 0 \\ 0 & 1 & 0 \end{vmatrix} = \vec{i}\begin{vmatrix} 0 & 0 \\ 1 & 0 \end{vmatrix} - \vec{j}\begin{vmatrix} 1 & 0 \\ 0 & 0 \end{vmatrix} + \begin{vmatrix} 1 & 0 \\ 0 & 1 \end{vmatrix} = 0\vec{i} - 0\vec{j} + 1\vec{k} = \vec{k}$$

By similar calculations, which we leave to you, we also have the relations $\vec{j}\times\vec{k} = \vec{i}$ and $\vec{k}\times\vec{i} = \vec{j}$:

$$\begin{aligned} \vec{i}\times\vec{j} &= \vec{k} \\ \vec{k}\times\vec{i} &= \vec{j} \\ \vec{j}\times\vec{k} &= \vec{i} \end{aligned} \tag{1.31}$$

Note that the vectors $\vec{i}$, $\vec{j}$, $\vec{k}$ are cyclically permuted:



(1.32)

in slots of the equation $\_ \times \_ = \_$

■

Proposition 1.13 (Algebraic Properties of the Cross Product) *Let $\vec{x}$, $\vec{y}$, $\vec{z}$ be vectors in $\mathbb{R}^3$ and let c be a real number. Then,*

(1) $\vec{x} \times \vec{y} = -\vec{y} \times \vec{x}$ *(anti-commutativity)*

(2) $\vec{x} \times (\vec{y} + \vec{z}) = \vec{x} \times \vec{y} + \vec{x} \times \vec{z}$
$(\vec{x} + \vec{y}) \times \vec{z} = \vec{x} \times \vec{z} + \vec{y} \times \vec{z}$ *(distributivity over addition)*

(3) $(c\vec{x}) \times \vec{y} = c(\vec{x} \times \vec{y}) = \vec{x} \times (c\vec{y})$ *(associativity of scalar multiplication and cross multiplication)*

Proof: Let us prove (1), and leave the rest as easy exercises. Let $\vec{x} = \langle a, b, c\rangle$ and $\vec{y} = \langle d, e, f\rangle$. Then,

$$\vec{y} \times \vec{x} = \begin{vmatrix} \vec{i} & \vec{j} & \vec{k} \\ d & e & f \\ a & b & c \end{vmatrix} = \vec{i}\begin{vmatrix} e & f \\ b & c \end{vmatrix} - \vec{j}\begin{vmatrix} d & f \\ a & c \end{vmatrix} + \begin{vmatrix} d & e \\ a & b \end{vmatrix} = (ce - bf)\vec{i} - (cd - af)\vec{j} + (bd - ae)\vec{k}$$

However, according to the calculation (1.30) above,

$$\vec{x} \times \vec{y} = (bf - ce)\vec{i} - (af - cd)\vec{j} + (ae - bd)\vec{k} = -\vec{y} \times \vec{x} \qquad \blacksquare$$

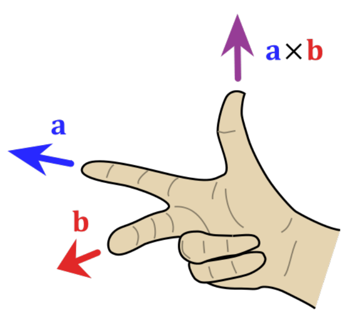
Proposition 1.14 *For any vectors $\vec{u}$, $\vec{v}$ in $\mathbb{R}^3$ we have the identity*

$$\boxed{|\vec{u} \times \vec{v}| = |\vec{u}||\vec{v}| \sin\theta} \tag{1.33}$$

and consequently

$$\boxed{\vec{u} \times \vec{v} = \big(|\vec{u}||\vec{v}| \sin\theta\big)\vec{n}} \tag{1.34}$$

where θ is the angle between the vectors and $\vec{n} = \frac{\vec{u}\times\vec{v}}{|\vec{u}\times\vec{v}|}$ is the unit vector in the direction of $\vec{u} \times \vec{v}$, determined by the **right-hand rule**, *illustrated in the following diagram,*[3]



Proof: Let $\vec{u} = \langle a, b, c\rangle$ and $\vec{v} = \langle d, e, f\rangle$. By the calculation (1.30) we have that $\vec{u} \times \vec{v} =$

[3] Which I got from the Wikipedia page on the cross product, http://en.wikipedia.org/wiki/Cross_product.

$\langle bf - ce,\ cd - af,\ ae - bd \rangle$, and consequently

$$\begin{aligned}
|\vec{u}|^2|\vec{v}|^2 \sin^2\theta &= |\vec{u}|^2|\vec{v}|^2(1 - \cos^2\theta) \\
&= |\vec{u}|^2|\vec{v}|^2 - |\vec{u}|^2|\vec{v}|^2 \cos^2\theta \\
&= |\vec{u}|^2|\vec{v}|^2 - (\vec{u} \cdot \vec{v})^2 \\
&= (a^2 + b^2 + c^2)(d^2 + e^2 + f^2) - (ad + be + cf)^2 \\
&= (a^2d^2 + b^2e^2 + c^2f^2 + b^2f^2 + c^2e^2 + c^2d^2 + a^2f^2 + a^2e^2 + b^2d^2) \\
&\qquad - (a^2d^2 + b^2e^2 + c^2f^2 + 2bcef + 2acdf + 2abde) \\
&= \big(b^2f^2 - 2bcef + c^2e^2\big) + \big(c^2d^2 - 2acdf + a^2f^2\big) + \big(a^2e^2 - 2abde + b^2d^2\big) \\
&\qquad + (a^2d^2 + b^2e^2 + c^2f^2) - (a^2d^2 + b^2e^2 + c^2f^2) \\
&= (bf - ce)^2 + (cd - af)^2 + (ae - bd)^2 \\
&= \langle bf - ce,\ cd - af,\ ae - bd \rangle \cdot \langle bf - ce,\ cd - af,\ ae - bd \rangle \\
&= |\vec{u} \times \vec{v}|^2
\end{aligned}$$

Taking the square root gives the result. ■

1.2.7 Interaction Between the Dot and Cross Products

Proposition 1.15 *Let* $\vec{a} = \langle a_1, a_2, a_3\rangle$, $\vec{b} = \langle b_1, b_2, b_3\rangle$, $\vec{c} = \langle c_1, c_2, c_3\rangle$ *be vectors in* $\mathbb{R}^3$. *Then,*

(1) $$\begin{aligned}\vec{a}\times(\vec{b}\times\vec{c}) &= (\vec{a}\cdot\vec{c})\vec{b}-(\vec{a}\cdot\vec{b})\vec{c}\\ (\vec{a}\times\vec{b})\times\vec{c} &= (\vec{a}\cdot\vec{c})\vec{b}-(\vec{c}\cdot\vec{b})\vec{a}\end{aligned} \qquad \textit{(vector triple product)}$$

(2) $$(\vec{a}\times\vec{b})\cdot\vec{c} = (\vec{c}\times\vec{a})\cdot\vec{b} = (\vec{b}\times\vec{c})\cdot\vec{a} = \begin{vmatrix} a_1 & a_2 & a_3\\ b_1 & b_2 & b_3\\ c_1 & c_2 & c_3\end{vmatrix} = \vec{a}\cdot(\vec{b}\times\vec{c}) = \vec{c}\cdot(\vec{a}\times\vec{b}) = \vec{b}\cdot(\vec{c}\times\vec{a})$$

(scalar triple product)

(3) $(\vec{a}\times\vec{b})\times\vec{c}+(\vec{c}\times\vec{a})\times\vec{b}+(\vec{b}\times\vec{c})\times\vec{a}=\vec{0}$ *(Jacobi identity)*

Proof: (1) This is a direct computation:

$$\begin{aligned}
\vec{a}\times(\vec{b}\times\vec{c}) &= \begin{vmatrix} \vec{i} & \vec{j} & \vec{k}\\ a_1 & a_2 & a_3\\ \begin{vmatrix} b_2 & b_3\\ c_2 & c_3\end{vmatrix} & -\begin{vmatrix} b_1 & b_3\\ c_1 & c_3\end{vmatrix} & \begin{vmatrix} b_1 & b_2\\ c_1 & c_2\end{vmatrix}\end{vmatrix}\\
&= \Big\langle a_2\begin{vmatrix} b_1 & b_2\\ c_1 & c_2\end{vmatrix} + a_3\begin{vmatrix} b_1 & b_3\\ c_1 & c_3\end{vmatrix},\\
&\qquad a_3\begin{vmatrix} b_2 & b_3\\ c_2 & c_3\end{vmatrix} - a_1\begin{vmatrix} b_1 & b_2\\ c_1 & c_2\end{vmatrix},\\
&\qquad\quad -a_1\begin{vmatrix} b_1 & b_3\\ c_1 & c_3\end{vmatrix} - a_2\begin{vmatrix} b_2 & b_3\\ c_2 & c_3\end{vmatrix}\Big\rangle\\
&= \big\langle a_2(b_1c_2-b_2c_1)+a_3(b_1c_3-b_3c_1),\\
&\qquad a_3(b_2c_3-b_3c_2)-a_1(b_1c_2-b_2c_1),\\
&\qquad\quad -a_1(b_1c_3-b_3c_1)-a_2(b_2c_3-b_3c_2)\big\rangle\\
&= \big\langle a_2b_1c_2-a_2b_2c_1+a_3b_1c_3-a_3b_3c_1,\\
&\qquad a_3b_2c_3-a_3b_3c_2-a_1b_1c_2+a_1b_2c_1,\\
&\qquad\quad -a_1b_1c_3+a_1b_3c_1-a_2b_2c_3+a_2b_3c_2\big\rangle\\
&\qquad +\langle a_1b_1c_1,\ a_2b_2c_2,\ a_3b_3c_3\rangle-\langle a_1b_1c_1,\ a_2b_2c_2,\ a_3b_3c_3\rangle\\
&= \big\langle (a_2c_2+a_3c_3)b_1-(a_2b_2+a_3b_3)c_1,\\
&\qquad (a_1c_1+a_3c_3)b_2-(a_1b_1+a_3b_3)c_2,\\
&\qquad\quad (a_1c_1+a_2c_2)b_3-(a_1b_1+a_2b_2)c_3\big\rangle\\
&\qquad +\langle a_1b_1c_1,\ a_2b_2c_2,\ a_3b_3c_3\rangle-\langle a_1b_1c_1,\ a_2b_2c_2,\ a_3b_3c_3\rangle\\
&= \Big\langle (a_1c_1+a_2c_2+a_3c_3)b_1-(a_1b_1+a_2b_2+a_3b_3)c_1,\\
&\qquad (a_1c_1+a_2c_2+a_3c_3)b_2-(a_1b_1+a_2b_2+a_3b_3)c_2,\\
&\qquad\quad (a_1c_1+a_2c_2+a_3c_3)b_3-(a_1b_1+a_2b_2+a_3b_3)c_3\Big\rangle\\
&= \big\langle (\vec{a}\cdot\vec{c})b_1-(\vec{a}\cdot\vec{b})c_1,\ \ (\vec{a}\cdot\vec{c})b_2-(\vec{a}\cdot\vec{b})c_2,\ \ (\vec{a}\cdot\vec{c})b_3-(\vec{a}\cdot\vec{b})c_3\big\rangle\\
&= (\vec{a}\cdot\vec{c})\vec{b}-(\vec{a}\cdot\vec{b})\vec{c}
\end{aligned}$$

The other expression in (1) follows from this one and the anti-commutativity of the cross product: $-(\vec{a}\times\vec{b})\times\vec{c} = \vec{c}\times(\vec{a}\times\vec{b}) = (\vec{c}\cdot\vec{b})\vec{a} - (\vec{c}\cdot\vec{a})\vec{b}$ so that $(\vec{a}\times\vec{b})\times\vec{c} = (\vec{a}\cdot\vec{c})\vec{b} - (\vec{c}\cdot\vec{b})\vec{a}$.

For (2), we simply compute, using some basic facts about determinants:

$$\begin{aligned}
(\vec{a}\times\vec{b})\cdot\vec{c} &= \left\langle \begin{vmatrix} a_2 & a_3 \\ b_2 & b_3 \end{vmatrix}, -\begin{vmatrix} a_1 & a_3 \\ b_1 & b_3 \end{vmatrix}, \begin{vmatrix} a_1 & a_2 \\ b_1 & b_2 \end{vmatrix} \right\rangle \cdot \langle c_1, c_2, c_3 \rangle \\
&= \begin{vmatrix} a_2 & a_3 \\ b_2 & b_3 \end{vmatrix} c_1 - \begin{vmatrix} a_1 & a_3 \\ b_1 & b_3 \end{vmatrix} c_2 + \begin{vmatrix} a_1 & a_2 \\ b_1 & b_2 \end{vmatrix} c_3 \\
&= \begin{vmatrix} a_1 & a_2 & a_3 \\ b_1 & b_2 & b_3 \\ c_1 & c_2 & c_3 \end{vmatrix} \\
&= \begin{vmatrix} c_1 & c_2 & c_3 \\ a_1 & a_2 & a_3 \\ b_1 & b_2 & b_3 \end{vmatrix} = (\vec{c}\times\vec{a})\cdot\vec{b} = \vec{b}\cdot(\vec{c}\times\vec{a}) \\
&= \begin{vmatrix} b_1 & b_2 & b_3 \\ c_1 & c_2 & c_3 \\ a_1 & a_2 & a_3 \end{vmatrix} = (\vec{b}\times\vec{c})\cdot\vec{a} = \vec{a}\cdot(\vec{b}\times\vec{c}) \\
&= \vec{c}\cdot(\vec{a}\times\vec{b})
\end{aligned}$$

since each of the other two determinants is obtained from the first by 2 row interchanges, which are equal to $(-1)^2$ times the first.

The Jacobi identity (3) follows from the vector triple product: $(\vec{a}\times\vec{b})\times\vec{c}+(\vec{c}\times\vec{a})\times\vec{b}+(\vec{b}\times\vec{c})\times\vec{a} = (\vec{a}\cdot\vec{c})\vec{b} - (\vec{a}\cdot\vec{b})\vec{c} + (\vec{c}\cdot\vec{b})\vec{a} - (\vec{c}\cdot\vec{a})\vec{b} + (\vec{b}\cdot\vec{a})\vec{c} - (\vec{b}\cdot\vec{c})\vec{a} = \vec{0}$. ■

1.3 Geometric Properties of the Dot and Cross Products

The formulas

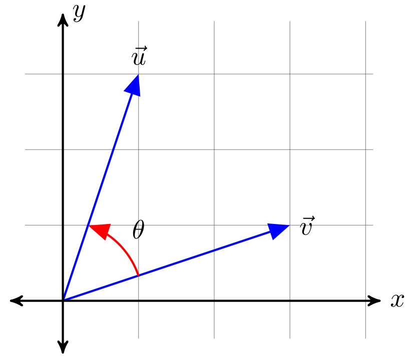
$$\vec{u} \cdot \vec{v} = |\vec{u}||\vec{v}| \cos\theta \tag{1.35}$$

$$|\vec{u} \times \vec{v}| = |\vec{u}||\vec{v}| \sin\theta \tag{1.36}$$

for all vectors $\vec{u}$ and $\vec{v}$ in $\mathbb{R}^3$ forming an acute angle θ, already contain significant geometric information. The purely algebraic definition of $\vec{u} \cdot \vec{v} = u_1v_1 + u_2v_2 + u_3v_3$, which a priori doesn't say anything about angles and orthogonality, turns out to in fact give precisely that information. Indeed, we can use purely algebraic information about $\vec{u}$ and $\vec{v}$, namely their magnitudes and dot product, to gain the important geometric information about the acute angle they form:

$$\theta = \cos^{-1}\left(\frac{\vec{u} \cdot \vec{v}}{|\vec{u}||\vec{v}|}\right) \tag{1.37}$$

(provided, of course, that neither vector is the zero vector $\vec{0} = \langle 0, 0, 0\rangle$, else we'd be dividing by 0).



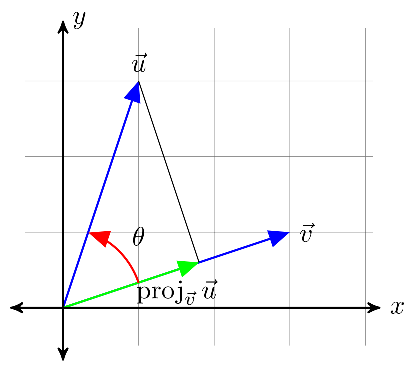
This fact motivates the following definition. We say two vectors $\vec{u}$ and $\vec{v}$ are **orthogonal** or **perpendicular** if $\vec{u} \cdot \vec{v} = 0$, and we denote this by

$$\boxed{\vec{u} \perp \vec{v}} \tag{1.38}$$

From equation (1.35) we immediately get that

$$\vec{u} \perp \vec{v} \iff \cos\theta = 0 \iff \theta = \frac{\pi}{2} \tag{1.39}$$

Next, consider the projection of the vector $\vec{u}$ onto the vector $\vec{v}$, that is, drop a perpendicular from the arrowhead of $\vec{u}$ onto the line containing $\vec{v}$,



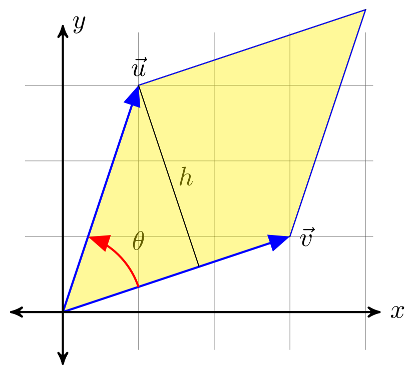
Observe that the length of the projection is clearly

$$\boxed{\mathrm{comp}_{\vec{v}}\,\vec{u} = |\,\mathrm{proj}_{\vec{v}}\,\vec{u}| = |\vec{u}|\cos\theta = |\vec{u}|\frac{|\vec{v}|}{|\vec{v}|}\cos\theta = \frac{\vec{u}\cdot\vec{v}}{|\vec{v}|}} \tag{1.40}$$

Therefore, if we give it a direction, namely the unit direction $\frac{\vec{v}}{|\vec{v}|}$ of $\vec{v}$, we get

$$\boxed{\mathrm{proj}_{\vec{v}}\,\vec{u} = (|\vec{u}|\cos\theta)\frac{\vec{v}}{|\vec{v}|} = \frac{\vec{u}\cdot\vec{v}}{|\vec{v}|^2}\vec{v}} \tag{1.41}$$

Next, consider the parallelogram formed by the vectors $\vec{u}$ and $\vec{v}$:



We know that its area is the length of its base times its height,

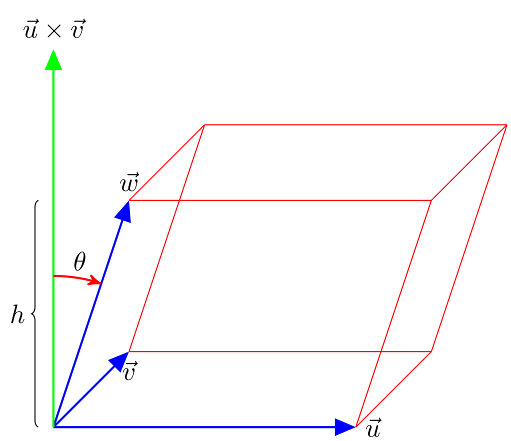
$$A = bh$$

Now, $b = |\vec{v}|$ and $h = |\vec{u}|\sin\theta$, so by equation

$$\boxed{A = bh = |\vec{v}||\vec{u}|\sin\theta = |\vec{u}\times\vec{v}|} \tag{1.42}$$

i.e. the area of the parallelogram determined by $\vec{u}$ and $\vec{v}$ is the length of their cross product!

Now consider a **parallelepiped** spanned by three vectors $\vec{u}$, $\vec{v}$, $\vec{w}$, the 3-dimensional analog of the parallelogram, a rectangular solid whose opposite sides are all parallel.



The volume is the area of the parallelogram spanned by $\vec{u}$ and $\vec{v}$ times the height h,

$$V = Ah = |\vec{u} \times \vec{v}|h$$

But note that h is the length of the projection of $\vec{w}$ onto $\vec{u} \times \vec{v}$,

$$h = |\operatorname{comp}_{\vec{u}\times\vec{v}} \vec{w}| = \frac{|\vec{w} \cdot (\vec{u} \times \vec{v})|}{|\vec{u} \times \vec{v}|} = \left|\vec{w} \cdot \left(\frac{\vec{u} \times \vec{v}}{|\vec{u} \times \vec{v}|}\right)\right| = |\vec{w}||\cos\theta|$$

Hence,

$$V = Ah = |\vec{u} \times \vec{v}|\frac{|\vec{w} \cdot (\vec{u} \times \vec{v})|}{|\vec{u} \times \vec{v}|} = |(\vec{u} \times \vec{v}) \cdot \vec{w}| \tag{1.43}$$

I.e., *the volume V is equal to the absolute value of the scalar triple product of $\vec{u}$, $\vec{v}$ and $\vec{w}$*, which, by our formulas for the scalar triple product from Proposition 1.15, can be computed using the determinant: if $\vec{u} = \langle u_1, u_2, u_3\rangle$, $\vec{v} = \langle v_1, v_2, v_3\rangle$ and $\vec{w} = \langle w_1, w_2, w_3\rangle$, then

$$\boxed{V = |(\vec{u} \times \vec{v}) \cdot \vec{w}| = \text{absolute value of } \begin{vmatrix} u_1 & u_2 & u_3 \\ v_1 & v_2 & v_3 \\ w_1 & w_2 & w_3 \end{vmatrix}} \tag{1.44}$$

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[1] Gallier, J. *Notes on Differential Geometry*, online notes at http://www.cis.upenn.edu/~jean/gbooks/manif.html, Version of August 2013.

[2] Knapp, A.W., *Basic Algebra*, Birkh auser, 2000.

[3] Munkres, J.R., *Analysis On Manifolds*, Westview Press, 1997.