The Determinant & Row Operations and Related Results

Thm. ~~3.3~~ 1 Using the recursive defn. of det, row operations have the following properties on det: $\forall A \in M_n(\mathbb{R})$,

(1) (type I, $E_1 = \begin{pmatrix} 1 & & & 0 \\ & \ddots & & \\ & & k & \\ 0 & & & 1 \end{pmatrix} \leftarrow$ ith row) scaling a row by $k \in \mathbb{R}$ ($k\vec{A}_i$)

$$\boxed{\det(E_1 A) = k \det A}$$

(2) (type II) ($E_2 A$: $c\vec{A}_i + \vec{A}_j$)

$$\boxed{\det(E_2 A) = \det A}$$

(3) (type III) ($E_3 A$: $\vec{A}_i \longleftrightarrow \vec{A}_j$)

$$\boxed{\det(E_3 A) = -\det A}$$

pf: We start w/ (3): By induction:

Base case, $n=2$: $A = \begin{pmatrix} \vec{A}_1 \\ \vec{A}_2 \end{pmatrix} = \begin{pmatrix} a & b \\ c & d \end{pmatrix}$,

$E = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}$

$$\Rightarrow \quad \det(EA) = \det\left(\begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}\begin{pmatrix} a & b \\ c & d \end{pmatrix}\right)$$
$$= \det\begin{pmatrix} c & d \\ a & b \end{pmatrix}$$
$$= bc - ad$$
$$= -(ad - bc) = -\det A.$$

Inductive step: suppose $\det(EA) = -\det A$ for all $n \geq 2$ and consider the $(n+1)$st case,

$A \in M_{n+1}(\mathbb{R})$, E swaps rows i & j $(E \in GL(n+1, \mathbb{R}))$.

There are 3 cases, ① $i, j \neq 1$

② ~~[illegible]~~ $i = 1$, $j = 2$

③ $i = 1$, $j > 2$

1 $i, j \neq 1$: Say $i < j$, else we can relabel.

$$\det(EA) = \det \begin{pmatrix} \vec{A}_1 \\ \vec{A}_i & \text{now } \vec{A}_j \\ \vdots \\ \vec{A}_j & \text{now } \vec{A}_i \\ \vdots \\ \vec{A}_m \end{pmatrix}$$

$$= \sum_{k=1}^{n} a_{1k} \det (\underset{\substack{ij \\ \text{swap}}}{E} A)_{1k}$$

$$= - \sum_{k=1}^{n} a_{1k} \det A_{1k} \leftarrow \left(\begin{array}{c}\text{by induction} \\ \text{hypothesis}\end{array}\right)$$

$$= - \det A$$

assuming 2

3 $i = 1, j > 2$:

$$\det \begin{pmatrix} \vec{A}_j \\ \vdots \\ \vec{A}_1 \\ \vdots \\ A_m \end{pmatrix} \overset{2}{=} - \det \begin{pmatrix} \vec{A}_2 \\ A_j \\ \vdots \\ A_1 \\ \vdots \\ A_m \end{pmatrix}$$

jth row

$$\overset{3}{=} \det \begin{pmatrix} \vec{A}_2 \\ \vec{A}_3 \\ A_j \\ \vdots \\ A_1 \\ \vdots \\ \vec{A}_m \end{pmatrix} = \cdots \overset{3}{=} (-1)^{j-2} \det \begin{pmatrix} \vec{A}_2 \\ \vdots \\ \vec{A}_{j-1} \\ \vec{A}_j \\ A_1 \\ \vdots \\ \vec{A}_m \end{pmatrix}$$

$$= (-1)^{j-1} \det \begin{pmatrix} \vec{A}_2 \\ \vdots \\ \vec{A}_{j-1} \\ \vec{A}_1 \\ \vec{A}_j \\ \vdots \\ \vec{A}_m \end{pmatrix} \begin{array}{l} \\ \\ \\ \leftarrow (j-1)\text{st pos.} \\ \leftarrow \text{in } j\text{th pos.} \\ \\ \\ \end{array}$$

$$= \underbrace{(-1)^{j-1}(-1)^{j}}_{=(-1)^{2j-1}=-1} \det \begin{pmatrix} \vec{A}_1 \\ \vdots \\ \vec{A}_m \end{pmatrix}$$

$$= -\det A$$

② $i=1,\ j=2$

~~Say $i<j$, else we can relabel.~~

The essential reason is that, while rows $\vec{A}_3, \ldots, \vec{A}_m$ stay the same, rows 1 & 2 are swapped, & because we scale a_{2j} by $(-1)^{j+1}$ & consequently a_{1k} by $(-1)^{k+1}$ if $k<j$ & by $(-1)^k$ if $k>j$, in $\det(EA)$, we need to compare w/ $\det A$:

On the other hand, in computing $\det A$, we scale a_{1k} by $(-1)^{k+1}$ & a_{2j} by $(-1)^{j}$ if $j>k$ & by $(-1)^{j+1}$ if $j<k$, & these give negatives of the 1st:

~~($j, k, E, \in \{1, 3, 4\}$ are distinct)~~

$$
\begin{aligned}
\det(\underset{\text{swap}}{E_{12}}A) &= \sum_{j=1}^{n} (-1)^{j+1} a_{2j} \Big(\sum_{j<k} (-1)^k a_{1k} A_{jk} \\
&\qquad + \sum_{j>k} (-1)^{k+1} a_{1k} A_{jk} \Big) \\
&= -\sum_{k=1}^{n} (-1)^{k+1} a_{1k} \Big(\sum_{j>k} (-1)^{j+1} a_{2j} A_{jk} \\
&\qquad + \sum_{j<k} (-1)^{j} a_{2j} A_{jk} \Big) \\
&= -\det A
\end{aligned}
$$

Consider:

$$\det \begin{pmatrix} \overset{+}{a_{21}} & \overset{-}{a_{22}} & \cdots & \overset{(-1)^{n+1}}{a_{2n}} \\ \overset{-}{a_{11}} & \overset{+}{a_{12}} & \cdots & \overset{(-1)^{n}}{a_{1n}} \\ \_\_\_ & \_\_\_ & \_\_\_ & \_\_\_ \\ \_\_\_ & \_\_\_ & \_\_\_ & \_\_\_ \end{pmatrix}$$

really boils down to

$$\boxed{\begin{aligned} & a_{21}\left(a_{12} - a_{13} + \cdots (-1)^{n} a_{1n}\right) \\ & \quad - a_{22}\left(\underline{a_{11}} - a_{13} + \cdots + (-1)^{n} a_{1n}\right) \\ & \qquad \ddots \\ & \qquad\qquad + (-1)^{n+1} a_{2n}\left(\underline{a_{11}} - a_{12} + \cdots + (-1)^{n} a_{1n}\right) \\ = {} & -a_{11}\left(a_{22} - a_{23} + \cdots + (-1)^{n+1} a_{2n}\right) \\ & \quad + a_{12}\left(a_{21} - a_{23} + \cdots + (-1)^{n} a_{2n}\right) \\ & \qquad \ddots \\ & \qquad + (-1)^{n} a_{1n}\left(a_{21} - a_{22} + \cdots + (-1)^{n+1} a_{2n}\right) \end{aligned}}$$

This proves (3) for all cases.

(2) <u>By induction:</u> <u>base case, $n=2$:</u>

(a) $A = \begin{pmatrix} a & b \\ c & d \end{pmatrix}$, $E = \begin{pmatrix} 1 & 0 \\ k & 1 \end{pmatrix}$ $(k\vec{A}_1 + \vec{A}_2)$

$$\det(EA) = \det\begin{pmatrix} a & b \\ c+ka & d+kb \end{pmatrix}$$

$$= a(d+kb) - b(c+ka)$$

$$= ad - bc + k\cancelto{0}{(ab - ab)}$$

$$= \det A$$

(b) $A = \begin{pmatrix} a & b \\ c & d \end{pmatrix}$, $E = \begin{pmatrix} 1 & k \\ 0 & 1 \end{pmatrix}$, $(k\vec{A}_2 + \vec{A}_1)$

$$\det(EA) = \det\begin{pmatrix} a+kc & b+kd \\ c & d \end{pmatrix}$$

$$= (a+kc)d - (b+kd)c$$

$$= ad - bc + k\cancelto{0}{(cd - cd)}$$

$$= \det A$$

Now suppose true for some $n \geq 2$ & consider the $(n+1)$st case: break it into cases: $c\vec{A}_i + \vec{A}_j$ w/

① $i, j \neq 1$

② $i = 1,\ j \neq 1$

③ $i \neq 1,\ j = 1$

<u>Case ①, $i, j \neq 1$</u>

$$\det \begin{pmatrix} \vec{A}_1 \\ c\vec{A}_i + \vec{A}_j \\ \vdots \\ \vec{A}_m \end{pmatrix} \leftarrow j\text{th row}$$

$$= \sum_{j=1}^{n} (-1)^{j+1} a_{1j} \underbrace{\det (EA)_{1j}}_{= \det A_{1j} \text{ by induction hyp.}}$$

$$= \sum_{j=1}^{n} (-1)^{j+1} a_{1j} \cdot \det A_{1j}$$

$$= \det A$$

<u>case ②: $i = 1, j \neq 1$</u> same, $\det \begin{pmatrix} \vec{A}_1 \\ \vdots \\ c\vec{A}_i + \vec{A}_j \\ \vdots \\ \vec{A}_m \end{pmatrix} = \det A$

<u>case ③, $i \neq 1, j = 1$:</u> $\det \begin{pmatrix} c\vec{A}_i + \vec{A}_1 \\ \vdots \\ \vec{A}_m \end{pmatrix} = -\det \begin{pmatrix} \vec{A}_2 \\ c\vec{A}_i + \vec{A}_1 \\ \vdots \\ \vec{A}_m \end{pmatrix}$

$$\overset{①}{=} -\det \begin{pmatrix} \vec{A}_2 \\ \vec{A}_1 \\ \vdots \\ \vec{A}_m \end{pmatrix} = \det \begin{pmatrix} \vec{A}_1 \\ \vdots \\ \vec{A}_m \end{pmatrix}$$

$$= \det A. \quad \text{QED}$$

(1) scaling row i By induction:

- $n=1$ / base case: $\det(ca) = ca = c\det(a)$
- inductive step: suppose true for some $n \geq 1$ & consider $(n+1)$st case: then cases:

① $c\vec{A}_1$: $\det\begin{pmatrix} c\vec{A}_1 \\ \vdots \\ \vec{A}_m \end{pmatrix}$

$= ca_{11}\det A_{11} - \cdots$

$= c\det A$

② $c\vec{A}_i,\ i>1$: $\det\begin{pmatrix} \vec{A}_1 \\ \vdots \\ c\vec{A}_i \\ \vdots \\ \vec{A}_m \end{pmatrix}$

$= a_{11}\underbrace{\det(EA)_{11}}_{=c\det A_{11} \text{ by induction hyp.}} - \cdots$

$= c\det A$

QED

* **Thm.** ~~3.2~~ 2 A upper triangular / diagonal $n\times n$ $\Rightarrow$

$$\det A = \prod_{i=1}^{n} a_{ii}$$

pf: easy.

Q: How to get A into upper triangular form, or diagonal ?

Thm. ~~3.5~~ 3 $\boxed{A \in GL(n,\mathbb{R}) \iff \det A \neq 0}$

pf:
$$\begin{aligned} \det(\operatorname{rref}(A)) &= \det(E_p \cdots E_1 A) \\ &= \pm c_1 \cdots c_k \det A, \quad c_i \neq 0 \end{aligned}$$

So since $A \in GL(n,\mathbb{R}) \iff \operatorname{rref}(A) = I$, we see that
$$\begin{aligned} \det(A) &= k \det(\operatorname{rref}(A)) \quad (k \neq 0) \\ &= \begin{cases} 0 & \text{if } A \notin GL(n,\mathbb{R}) \\ k \neq 0, & \text{if } A \in GL(n,\mathbb{R}). \end{cases} \end{aligned}$$

QED

Cor. ~~3.6~~ $\forall A \in M_n(\mathbb{R})$, $\boxed{\operatorname{rank} A = n \iff \det A = 0}$

Thm. ~~37~~ 4 $\quad \det(AB) = (\det A)(\det B)$

i.e. $\quad \det : (M_n(\mathbb{R}), \circ) \longrightarrow (\mathbb{R}, \circ)$

is a monoid homomorphism, and

$$GL(n,\mathbb{R}) = \det^{-1}(\mathbb{R} \setminus \{0\})$$

Pf: (1) If $A \notin GL(n,\mathbb{R})$, then $\det A = 0$, but also $\det(AB) = 0$, else $AB \in GL(n,\mathbb{R})$, & since $(AB)^{-1} = B^{-1}A^{-1}$, this should imply a contradiction: $A \in GL(n,\mathbb{R})$, but if $(AB)^{-1}$ exists, then

$$B(AB)^{-1} = A^{-1}$$

exists too, since $A(B(AB^{-1})) = (AB)(AB)^{-1} = I$. Thus, ~~Conversely, if det~~ if $A \notin GL(n,\mathbb{R})$, then

$$\det(AB) = 0 = 0 \cdot \det B = (\det A)(\det B)$$

② If $A \in GL(n,\mathbb{R})$, then,

$$\det(E_i E_j) = (\det E_i)(\det E_j)$$

by Thm 3.3, we will have $A = E_1 \cdots E_p I$

$$\Rightarrow AB = E_1 \cdots E_p B$$

$$\begin{aligned}\det(AB) &= (\det(E_1)) \cdots (\det E_p)(\det B) \\ &= (\det(E_1 \cdots E_p)) \det B \\ &= (\det A)(\det B).\end{aligned}$$

QED

Cor. 3-8 $\quad \det(A^{-1}) = \dfrac{1}{\det A}$ if $A \in GL(n,\mathbb{R})$.

pf: $\underset{\neq 0}{(\det A)}(\det A^{-1}) = \det(AA^{-1})$
$= \det I$
$= 1$

$$\Rightarrow \det(A^{-1}) = \frac{1}{\det A}.$$

QED