Project 4: Polynomial Approximations of Continuous Functions

Write up proofs for the following **theorems**:

(1) (see Exercise 6.6.3 and Theorem 6.6.3) **Taylor's Theorem:** Let $f \in C^{n+1}[a,b]$, and let $x_0 \in (a,b)$. Then, for all $x \in (a,b)$ with $x \neq x_0$ there is some x^* between x_0 and x such that

$$\begin{aligned} f(x) &= f(x_0) + f'(x_0)(x-x_0) + \frac{f''(x_0)}{2!}(x-x_0)^2 + \cdots + \frac{f^{(n)}(x_0)}{n!}(x-x_0)^n + R_n(x^*) \\ &= \sum_{k=0}^{n} \frac{f^{(k)}(x_0)}{k!}(x-x_0)^k + R_n(x^*) \end{aligned}$$

where

$$R_n(x^*) = \frac{f^{(n+1)}(x^*)}{(n+1)!}(x-x_0)^{n+1} \qquad \text{(\textit{Lagrange remainder})}$$

$$\left.\begin{aligned} &= \int_x^{x_0} \frac{f^{(n+1)}(t)}{n!}(x-t)^n dt, \quad \text{if } x < x_0 \\ &= \int_{x_0}^{x} \frac{f^{(n+1)}(t)}{n!}(x-t)^n dt, \quad \text{if } x_0 < x \end{aligned}\right\} \qquad \text{(\textit{Cauchy remainder})}$$

Some Remarks:

- *Taylor's Theorem concerns polynomial approximations of C^n functions on closed bounded intervals $[a,b]$ in a neighborhood of a point.*
- *Let $p(x) = \sum_{k=0}^{n} \frac{f^{(k)}(x_0)}{k!}(x-x_0)^k$ be the n**th Taylor polynomial** approximating f on $[a,b]$. The Lagrange form of the remainder $R(x^*)$ can be used to find a bound on error size of the approximation,*

$$E(x) = |f(x) - p(x)| \leq \max_{a \leq x^* \leq b} |R(x^*)| = \frac{MN}{(n+1)!}$$

where $M = \max_{a \leq x \leq b} |f^{n+1}(x)|$ and $N = \max\{x-a,\ b-x\}$.

(2) (see Theorem 6.7.1) **Weierstrass Approximation Theorem:** For every continuous function $f \in C]a,b]$ and for any $\varepsilon > 0$ there exists a polynomial $p \in \mathbb{R}[x]$ approximating f uniformly on $[a,b]$ within ε,

$$|f(x) - p(x)| < \varepsilon, \quad \forall x \in [a,b]$$

(3) **Lagrange Interpolating Polynomials**

(a) **Discussion and Definitions:** Given distinct constants $c_0, c_1, \dots, c_n \in \mathbb{R}$, we can define $n+1$ distinct polynomials that each have all except one of the $c_0, \dots, c_n$ as roots, and such that they are linearly independent as vectors in $\mathbb{R}_n[x]$ (the vector space of polynomials of degree $\leq n+1$). This gives us a special basis for $\mathbb{R}_{n+1}[x]$, one which allows the construction of an $n+1$-degree polynomial passing through the points $(c_0, b_0), \dots, (c_n, b_n) \in \mathbb{R}^2$. The polynomials $\ell_0(x), \dots, \ell_n(x) \in \mathbb{R}_n[x]$ associated with the $c_0, \dots, c_n$ are given by

$$\ell_i(x) = \prod_{\substack{k=0 \\ k \neq i}}^{n} \frac{x - c_k}{c_i - c_k}$$

As a consequence of this definition we have that

$$\ell_i(c_j) = \begin{cases} 0, & \text{if } i \neq j \\ 1, & \text{if } i = j \end{cases}$$

This property of the polynomials means that $\beta = \{\ell_0, \ell_1, \dots, \ell_n\}$ is a basis for $\mathbb{R}_n[x]$ (the real vector space of polynomials of degree $\leq n$)–see proof below. As a result, if we specify a set of $n+1$ constants in $b_0, b_1, \dots, b_n \in \mathbb{R}$, the polynomial

$$\boxed{L(x) = \sum_{i=0}^{n} b_i \ell_i(x)}$$

is the unique polynomial in $\mathbb{R}_n[x]$ such that $L(c_j) = b_j$ for all $j = 0, 1, \dots, n$, called the n**th Lagrange polynomial** interpolating the c_i. The process outlined above of finding the polynomial $L(x)$ satisfyingt $L(c_j) = b_j$ for all $j = 0, 1, \dots, n$ is called **Lagrange interpolation**.

Now for the proof that $\beta = \{\ell_0, \ell_1, \dots, \ell_n\}$ is a basis for $\mathbb{R}_n[x]$:

Proof: First, we prove that $\beta = \{\ell_0, \ell_1, \dots, \ell_n\}$ is linearly independent in $\mathbb{R}_n[x]$. That is, letting $p_0(x) \equiv 0$ denote the zero polynomial, we prove that

$$\forall a_0, \dots, a_n \in \mathbb{R}, \ (a_0\ell_0 + \cdots + a_n\ell_n = p_0 \implies a_0 = \cdots = a_n = 0)$$

Well, the premise, when spelled out more fully like this:

$$\sum_{j=0}^{n} a_j \ell_j(x) = p_0(x) = 0$$

s holds true for *all* $x \in \mathbb{R}$. This gives us the idea of plugging in c_j for x:

$$\begin{aligned} a_j &= a_j \ell_j(c_j) \\ &= a_0 \ell(c_j) + a_1 \ell_1(c_j) + \cdots + a_n \ell_n(c_j) \\ &= p_0(c_j) \\ &= 0 \end{aligned}$$

This is true for each $j = 0, 1, \ldots, n$. We conclude that $a_0 = \cdots = a_n = 0$ and therefore β is linearly independent.

We would normally next show that $\beta = \{\ell_0, \ell_1, \ldots, \ell_n\}$ is panning. But since $\dim\big(\mathbb{R}_n[x]\big) = n+1$ and $|\beta| = n+1$, it follows from linear algebra β is a basis for $\mathbb{R}_n[x]$. Consequently, every n-th (or lower) degree polynomial in $\mathbb{R}_n[x]$ is a unique linear combination of polynomials in β, so that if $L \in \mathbb{R}_n[x]$, then $\exists b_0, \ldots, b_n \in \mathbb{R}$ such that

$$\boxed{L = b_0 \ell_1 + \cdots + b_n \ell_n}$$

(b) **Theorem (Lagrange Interpolation):** Suppose we have $f \in C^{n+1}[a, b]$ and use the y-values of f at distinct points $x_0, x_1, \ldots, x_n \in [a, b]$ instead of the constants b_i in the construction of the Lagrange interpolating polynomial L. By the above discussion

$$L(x) = f(x_0)\ell_0(x) + f(x_1)\ell_1(x) + \cdots + f(x_n)\ell_n(x)$$

is the unique polynomial in $\mathbb{R}_n[x]$ passing through the points $(x_i, f(x_i))$, and thus interpolating f. We claim that

$$f(x) = L(x) + \frac{f^{(n+1)}(\xi(x))}{(n+1)!} \prod_{i=0}^{n} (x - x_i)$$

Consequently, a bound for the error $|f(x) - L(x)|$ may be found by maximizing $\frac{f^{(n+1)}(\xi(x))}{(n+1)!} \prod_{i=0}^{n} (x - x_i)$ on $[a, b]$.

Look at Theorem 3.3 in Burden and Faires, Numerical Analysis, 9th Edition for a nice proof.