Project 1: Cantor Set

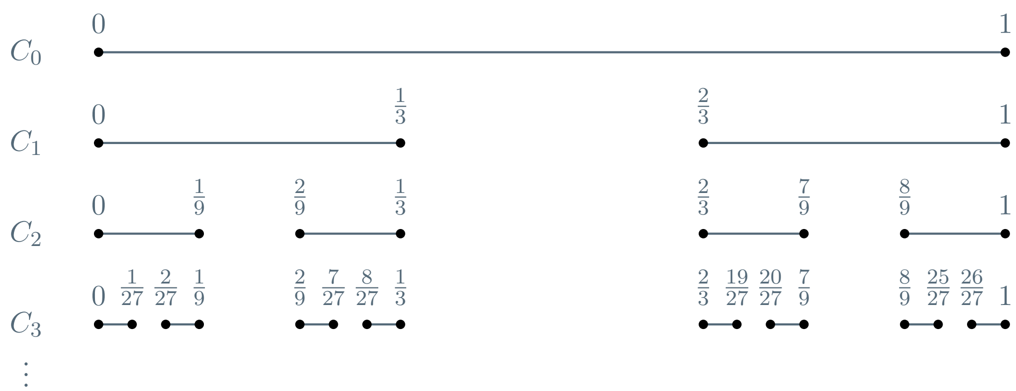
Definition 1 The **Cantor set** is defined as the limit of a sequence of nested closed subsets of $[0,1]$ as follows: Let $C_0 = [0,1]$, then let $C_1 = [0,1] \backslash (\frac{1}{3}, \frac{2}{3})$, and let $C_2 = C_1 \backslash \big((\frac{1}{9}, \frac{2}{9}) \cup (\frac{7}{9}, \frac{8}{9}) \big)$, and so on, so that $C_1 = [0, \frac{1}{3}] \cup [\frac{2}{3}, 1]$, $C_2 = [0, \frac{1}{9}] \cup [\frac{2}{9}, \frac{1}{3}] \cup [\frac{2}{3}, \frac{7}{9}] \cup [\frac{8}{9}, 1]$. Continuing in this manner, C_k is obtained from C_{k-1} by removing the middle third of each closed subinterval of C_{k-1}. Then $C_k \searrow C$, where

$$C := \bigcap_{k=1}^{\infty} C_k \tag{0.1}$$

is the Cantor set. An explicit formula for the Cantor set, in terms of the subsets of $[0,1]$ removed, is

$$C = [0,1] \backslash \Big(\bigcup_{m=1}^{\infty} \bigcup_{k=0}^{3^{m-1}-1} \Big(\frac{3k+1}{3^m}, \frac{3k+2}{3^m} \Big) \Big) \tag{0.2}$$

Its construction can be visualized as follows:



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Write up proofs for the following **theorems**:

1. The Cantor set C is **compact**.
2. The Cantor set C has **empty interior**, $C^{\circ} = \varnothing$, and has **total length** 0. In fact, in combination with (1) this shows that C is **nowhere dense**, that is $(\overline{C})^{\circ} = \varnothing$.
3. The Cantor set C is **perfect**, that is $C = L(C)$. Hence, C has **no isolated points**, $I(C) = \varnothing$.
4. The Cantor set C is **uncountable**, that is C is in bijective correspondence with $\mathbb{R}$.
5. The number $\frac{1}{4} \in C$, but $\frac{1}{4}$ is not an endpoint of any of the sets C_k. Thus, the Cantor set C doesn't consist only of endpoints of the C_k.